MATH53MAY7

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Review of 3-d Calculus:

Given an open domain $D \subseteq \mathbb{R}^3$, we have two <u>sequences</u> and a <u>pairing</u>:

$$0 \xrightarrow{d} \mathbb{R} \xrightarrow{d} \underset{\text{functions}}{C^\infty(D)} \xrightarrow[\text{grad}]{d} \underset{\Omega^1(D)}{\text{vector fields on } D} \xrightarrow[\text{curl}]{d} \underset{\Omega^2(D)}{\text{vector fields on } D} \xrightarrow[\text{div}]{d} C^\infty(D) \xrightarrow{d} 0$$

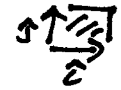
$$0 \leftarrow \mathbb{Z} \xleftarrow{\partial} \text{(sets of) oriented points} \xleftarrow{\partial} \text{oriented paths} \xleftarrow{\partial} \text{oriented surfaces} \xleftarrow{\partial} \text{oriented volumes} \leftarrow 0$$

$$\mathbb{R} \ni \qquad \sum_{x \in X} f(x)(\pm 1) \qquad \int_\gamma \vec{F} \cdot d\vec{r} \qquad \iint_S \vec{F} \cdot d\vec{S} \qquad \iiint_V f \, dV$$

The "pairing" takes one thing (function or vector field) from the top row and one thing (set of oriented k-dimensional thingies) from the next row, and integrates to get a real number.

The $\rightarrow$ arrows in the top row are all called "d", and are various kinds of "derivatives". ~~Here's~~ The "0th d" is $0 \mapsto 0 \in \mathbb{R}$. The "1st d" is $a \mapsto \{\text{constant function } f(x,y,z) = a\}$. The last d sends all functions to 0. Here is a general formula for the rest:

- Take the gradient of each term, even if it's multiplied by $\hat{\imath}, \hat{\jmath}, \hat{k}$, etc.
- Multiply in the "grade-school" way: the

product $\hat{i}\times\hat{j}$ is the ~~area~~ rectangle  with orientation from the right-hand-rule. The triple product is $\hat{i}\times\hat{j}\times\hat{k} = 1$. For example:

$$f_1\hat{i}+f_2\hat{j}+f_3\hat{k} \overset{d}{\longmapsto} \left(\frac{\partial f_1}{\partial x}\hat{i}+\frac{\partial f_1}{\partial y}\hat{j}+\frac{\partial f_1}{\partial z}\hat{k}\right)\times\hat{i}$$

$$+\left(\frac{\partial f_2}{\partial x}\hat{i}+\frac{\partial f_2}{\partial y}\hat{j}+\frac{\partial f_2}{\partial z}\hat{k}\right)\times\hat{j}$$

$$+\left(\frac{\partial f_3}{\partial x}\hat{i}+\frac{\partial f_3}{\partial y}\hat{j}+\frac{\partial f_3}{\partial z}\hat{k}\right)\times\hat{k}$$

$$= 0 + \frac{\partial f_1}{\partial y}\hat{j}\times\hat{i} + \frac{\partial f_2}{\partial z}\hat{k}\times\hat{i}$$

$$+\frac{\partial f_2}{\partial x}\hat{i}\times\hat{j} + 0 + \frac{\partial f_2}{\partial z}\hat{k}\times\hat{j}$$

$$+\frac{\partial f_3}{\partial x}\hat{i}\times\hat{k} + \frac{\partial f_3}{\partial y}\hat{j}\times\hat{k} + \cancel{\frac{\partial f_3}{\partial z}\hat{k}\times\hat{k}} \to 0$$

$$= \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\hat{i}\times\hat{j} + \ldots$$

since the area of the $\hat{i}$-by-$\hat{i}$ box is 0.

Exercise: this was the third d; work out the next one.
On the second row, ∂ takes the boundary, oriented with Right-hand rule.
For each row, we have the

Zoop-zoop theorem:

$d^2=0$, $\partial^2=0$. I.e. following arrows twice goes to 0.

If the domain D is contractible (e.g. $D=\mathbb{R}^3$), then the converse of the zoop-zoop theorem holds:

Zoop-Zoop converse:

If $D = \mathbb{R}^3$ and a thing has $\partial(\text{thing}) = 0$, then there is some "antiderivative" Whatsit so that $\partial(\text{Whatsit}) = \text{thing}$.

The failure of this converse to hold for other Ds measures the topology of D. (By the way: $\partial($  $) = \ominus \; \oplus$, and $\partial(\oplus) = +1$, $\partial(\ominus) = -1$, so $\partial(\ominus \; \oplus) = -1 + 1 = 0$.)

We also have

Generalized Stokes' theorem:

Given an F in the top row, and an S in the second row one column to the right, the two ways of moving and pairing are the same:

$$\int_{\partial S} F = \int_S \partial F$$

In this class we proved every version (each column) of all these theorems, for 2- and 3-dimensional calculus. I hope you can believe that it continues to 4- and higher-dimensional settings.

4-D CALCULUS AND ELECTROMAGNETISM

In 4-D, the top row is

functions $\xrightarrow{d}$ vector fields $\xrightarrow{d}$ "2-forms" $\xrightarrow{d}$ vector fields $\xrightarrow{d}$ functions

Use variables t, x, y, z for $\mathbb{R}^4$, and unit vectors $\hat{h}, \hat{\imath}, \hat{\jmath}, \hat{k}$. Then, e.g., the first d is

$$\text{grad}(f(t,x,y,z)) = \frac{\partial f}{\partial t}\hat{h} + \frac{\partial f}{\partial x}\hat{\imath} + \frac{\partial f}{\partial y}\hat{\jmath} + \frac{\partial f}{\partial z}\hat{k}$$

$$= \left(\frac{\partial f}{\partial t}, \vec{\nabla} f\right)$$

where $\vec{\nabla}$ is the three-dimensional gradient.

<u>Exercise:</u> the second d is

$$d(f_0, \vec{f}) = d\left(f_0\hat{h} + f_1\hat{\imath} + f_2\hat{\jmath} + f_3\hat{k}\right) = \ldots$$

$$= \left(\left(\frac{\partial \vec{f}}{\partial t} - \vec{\nabla} f_0\right), \vec{\nabla} \times \vec{f}\right)$$

<u>Exercise:</u> The third d is

$$d(\vec{E}, \vec{B}) = \left(\vec{\nabla} \cdot \vec{B}, \frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \vec{E}\right)$$

If $\vec{E}$ is the electric field and $\vec{B}$ is the magnetic field, then <u>Maxwell's Equations</u> say

$$d(\vec{B}, \vec{E}) = (\rho, \vec{J}) = (\text{charge density, current density}).$$

Since $d^2 = 0$, we have $d(\rho, \vec{J}) = 0$, i.e. <u>Conservation of charge</u>.
Then $d(\vec{E}, \vec{B}) = 0$ by Maxwell, so $(\vec{E}, \vec{B}) = d(V, \vec{A})$ by zoop-zoop converse: i.e. there is an <u>electromagnetic potential</u>.