Math 53 9 April 08

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① Let $\vec{r}(t)$ be the path

$$\vec{r}(t) = \left(e^t \sin \pi t,\ t^2 \cos \pi t\right) = \left(x(t), y(t)\right)$$

As $t$ goes from $0$ to $1$, $\vec{r}$ goes from $(0,0)$ to $(0,-1)$.

~~With it out~~

Recall that we define $\displaystyle\int_{\{\vec{r}\}} \vec{v}(\vec{r}) \cdot d\vec{r}$ as $\displaystyle\int_{t=0}^{1} \vec{v}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$

for a vector field $\vec{v}(x,y)$. Without doing much work, compute the following integrals:

(a) $\int \hat{\imath} \cdot d\vec{r}$

(b) $\int \hat{\jmath} \cdot dr$

(c) $\int x\hat{\imath} \cdot d\vec{r}$

~~(d) $\int x\hat{\jmath}$~~

(d) $\int e^x \hat{\imath} \cdot d\vec{r}$

(e) $\int e^y \hat{\jmath} \cdot dr$

(f) $\int (x\hat{\jmath} + y\hat{\imath}) \cdot d\vec{r}$ Hint: what is $\vec{\nabla}(xy)$?

(g) $\int \vec{\nabla}\left[\ln(1+x^2) + ye^x + \arctan(y)\right] \cdot d\vec{r}$

② Recall that $\vec{\nabla}\left(\arctan \frac{y}{x}\right) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$.

(a) Integrate $\vec{\nabla}\left(\arctan \frac{y}{x}\right)$ along the path $x(t) = 1$, $y(t) = t$ for $t$ ranging from $0$ to $1$.

(b) Integrate $\vec{\nabla}(\arctan \frac{y}{x})$ along the path

$$x(t) = 2 + \cos t$$
$$y(t) = 3 + \sin t$$

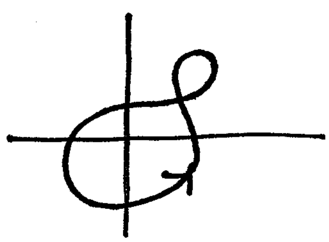
for $t$ ranging from $0$ to $2\pi$.

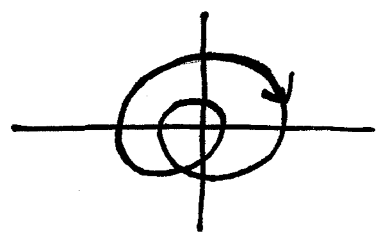
(c) Integrate $\vec{\nabla}(\arctan \frac{y}{x})$ around the circle

$$x(t) = \cos t$$
$$y(t) = \sin t$$

for $t$ ranging from $0$ to $2\pi$. Why is this not a contradiction to the fundamental theorem of calculus?

(d) Integrate $\vec{\nabla}(\arctan \frac{y}{x})$ around the following closed loops:





③ Decide which of the following fields are conservative:

(a) $\vec{\nabla}\left(\frac{1}{x^2+y^2}\right)$

(b) $\langle x^2-y^2, 2xy \rangle$

(c) $\langle e^x \cos y, e^x \sin y \rangle$

(d) $\langle y, y^2 \rangle$

(e) $\langle f(x), g(y) \rangle$

(f) $\langle g(y), f(x) \rangle$

For (e) and (f): for what functions $f, g$ are the fields conservative?