Midterm I

1. The plane curve C has parametric representation

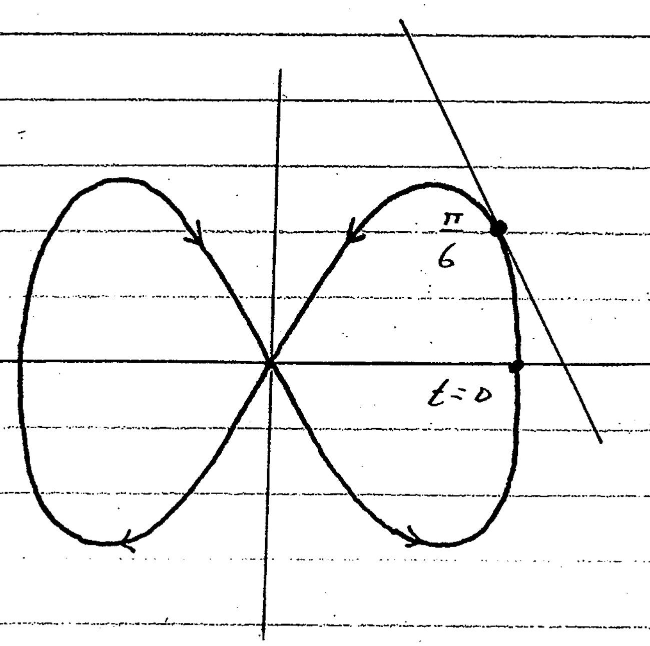
$x = \cos t, \quad y = \sin 2t, \quad 0 \le t \le 2\pi.$

Compute the equation of the tangent line at the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$. Sketch the curve C and the tangent line.

Sol'n:

$$m = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos 2t}{-\sin t} = \frac{-2\cos\frac{\pi}{3}}{\sin\frac{\pi}{6}} = \frac{-2\left(\frac{1}{2}\right)}{\frac{1}{2}} = -2.$$

$$y - \frac{\sqrt{3}}{2} = (-2)\left(x + \frac{\sqrt{3}}{2}\right)$$



2. A line has the vector equation

$$\underline{r}(t) = \underline{i} + \underline{j} + \underline{k} + (\underline{j} + \underline{k})t.$$

Find the point p on this line so that the displacement vector from the origin to p is perpendicular to the line.

Sol'n: p has $t = t_c$ determined by

$$\underline{r}(t_c) \cdot (\underline{j} + \underline{k}) = 0$$

$$\{\underline{i} + \underline{j} + \underline{k} + (\underline{j} + \underline{k})t_c\} \cdot (\underline{j} + \underline{k}) = 0$$

$$2 + 2t_c = 0 \Rightarrow t_c = -1.$$

$$\underline{r}(t_c) = \underline{i} + \underline{j} + \underline{k} + (\underline{j} + \underline{k})(-1) = \underline{i}.$$

p: $(1, 0, 0)$

3. Compute the equation of the plane which passes through the point $(1, 1, 1)$ and contains the line whose parametric equations are

$$x = t, \quad y = 2t, \quad z = 3t.$$

<u>Soltn</u> 3 points $O: (0, 0, 0)$ $\quad P: (1, 2, 3)$ $\quad Q: (1, 1, 1)$

$$\overrightarrow{OP} = \underline{i} + 2\underline{j} + 3\underline{k} \qquad \overrightarrow{OQ} = \underline{i} + \underline{j} + \underline{k}$$

Normal

$$\underline{n} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -\underline{i} + 2\underline{j} - \underline{k}.$$

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k} \quad \text{so} \quad \underline{r} \cdot \underline{n} = \boxed{-x + 2y - z = 0}$$

4. The space curve C has the vector equation

$$\underline{r}(t) = (\underline{i}+\underline{j})\cos t + (\underline{i}+\underline{k})\sin t, \quad 0 \le t \le 2\pi.$$

What points on C are farthest from the origin?

Soltn: $\underline{r}(t) = (\cos t + \sin t)\underline{i} + \underline{j}\cos t + \underline{k}\sin t$

$$|\underline{r}|^2 = (\cos t + \sin t)^2 + \cos^2 t + \sin^2 t =$$

$$2(1 + \cos t \sin t) = 2 + \sin 2t$$

$\uparrow$

max for $t = \frac{\pi}{4}, \frac{5\pi}{4}$

$$\underline{r}\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}}\underline{i} + \frac{1}{\sqrt{2}}\underline{j} + \frac{1}{\sqrt{2}}\underline{k}$$

point P: $\left(\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. $\quad t = \frac{5\pi}{4} \Rightarrow \left(-\sqrt{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$