Math 1B Handout: Estimating Series

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/08Summer1B/

Monday, 11 August 2008

Approximating Functions by Polynomials

Let's say that we have a function $f(x)$ with Taylor series $\sum_{k=0}^{\infty} f^{(k)}(0)x^k/k!$. Let's add up only the first n terms, and define

$$T_n(x) \stackrel{\text{def}}{=} f(0) + \frac{f'(0)}{1}x + \frac{f''(0)}{2}x^2 + \cdots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1} + \frac{f^{(n)}(0)}{n!}x^n = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}x^k$$

If x is inside the radius of convergence of the power series, then $f(x) = \sum_{k=0}^{\infty} f^{(k)}(0)x^k/k! = \lim_{n\to\infty} T_n(x)$. How fast does this limit converge? Let's define the difference between $f(x)$ and $T_n(x)$ to be the "error"

$$R_n(x) \stackrel{\text{def}}{=} f(x) - T_n(x)$$

The strong form of Taylor's theorem gives a bound on the size of this error. If $a > 0$ and $|x| < a$, then

$$|R_n(x)| \leq \sup_{|z|\leq a} \frac{\left|f^{(n+1)}(z)\right|}{(n+1)!}a^n \stackrel{\text{def}}{=} E_n \tag{1}$$

The word "sup" is short for "supremum", and means that we take the z with $|z| \leq a$ that makes the $(n+1)$th derivative largest.

1. Use equation (1) to estimate the size of the error $\left|\sin(x) - (x - x^3/6)\right|$ when $|x| \leq 1$.

2. What are the derivatives of $\sin(x)$? Use this to bound $\sup_{|z|\leq a}\left|f^{(n+1)}(z)\right|$. Hence, for what a can you be sure that $\left|\sin(x) - (x - x^3/6 + x^5/120)\right| \leq 0.001$ for all $x \leq a$? For what odd number n can you be sure that $\left|\sin(x) - (x - x^3/6 + \cdots \pm x^n/n!)\right| \leq 0.001$ for $x \leq 10$?

3. Use equation (1) to estimate the error in the approximation
$$e^3 \approx 1 + 3 + 3^2/2 + 3^3/6 + 3^4/4! + 3^5/5!$$

4. Let's assume that the ratio test works for the power series $\sum_{k=0}^{\infty} f^{(k)}(0)x^k/k!$ — i.e. assume that the limit L of the sequence $(n+1)f^{(n)}(0)/f^{(n+1)}(0)$ exists. What is this limit, in terms of the power series? Show that if $a \geq L$, then the sequence E_n in equation (1) tends to $+\infty$. (Hint: take $z = 0$, and think about the ratio test.)

Estimating sums by integrals

Let a_n be a positive decreasing sequence that tends to 0. Then we can find a positive decreasing function $f(x)$ so that $a_n = f(n)$ for every positive integer n. Then the integral test tells us that

$$\int_1^\infty f(x)\,dx \le \sum_{n=1}^\infty a_n \le \int_0^\infty f(x)\,dx$$

This estimates the series $\sum a_n$ by an improper integral: in particular, the integral converges if and only if the sum does.

Instead of estimating the entire series by an integral, we could just estimate the tail. The integral test gives

$$\int_{n+1}^\infty f(x)\,dx \le \sum_{k=n+1}^\infty a_k \le \int_n^\infty f(x)\,dx$$

But the middle term is the error between the infinite sum s and the nth partial sum s_n:

$$\sum_{k=n+1}^\infty a_k = \left(\sum_{k=0}^\infty a_k\right) - \left(\sum_{k=0}^n a_k\right)$$

Hence we have two estimates:

$$\sum_{k=0}^\infty a_k = \sum_{k=0}^n a_k + \left(\text{error} \le \int_n^\infty f(x)\,dx\right) \tag{2}$$

$$\sum_{k=0}^\infty a_k = \sum_{k=0}^n a_k + \int_{n+1}^\infty f(x)\,dx + \left(\text{error} \le \int_n^{n+1} f(x)\,dx \le a_n\right) \tag{3}$$

1. Use equation (2) to estimate the error $\left|\pi^2/6 - (1 + 1/4 + 1/9 + 1/16 + 1/25)\right|$. For what n is the error $\left|\pi^2/6 - (1 + 1/4 + \cdots + 1/n^2)\right|$ less than 0.0001?

2. Use equation (3) to estimate $\pi^2/6$ to within an error of 0.0001. I.e. write a finite sum of fractions that gets within that bound (one of your fractions should come from the integral term).

3. Let $\sum a_n$ be an absolutely convergent series (where some of the terms might be negative). Show that

$$\left|\sum_{k=n}^\infty a_k\right| \le \sum_{k=n}^\infty |a_k|$$