Math 1B Handout: Estimating Integrals by Sums

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Given a continuous function $f(x)$ and an interval $[a, b]$, we defined long ago the integral $\int_a^b f(x)dx$: Riemann says to divide the integral into lots of small pieces, and add up the heights of the function on each piece, multiplying by the thickness of a piece. In Riemann's definition, we take the limit as "lots" becomes "infinitely many", and define the final limit as the *integral*.

However, taking a limit directly requires doing infinitely many calculations. Sometimes we can use tricks to take limits (e.g. the limit of a rational function), but otherwise we should approximate the limit by simply picking a large number. But then we have to decide how to define the "height" of each rectangle in Riemann's formalism. Here are a few good ways. Each has an associated error formula:

Right- or Left-Hand Rule Divide $[a, b]$ into n equal pieces. Let $\Delta x = (b - a)/n$ be the width of the piece, and $x_i = a + i\Delta x$ be the ith division (so $x_0 = a$ and $x_n = b$). To each piece, associate the left- or right-endpoint heights:

$$\int_a^b f(x)\,dx \approx L_n \stackrel{\text{def}}{=} [f(x_0) + f(x_1) + \cdots + f(x_{n-2}) + (x_{n-1})]\,\Delta x = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x$$

$$\int_a^b f(x)\,dx \approx R_n \stackrel{\text{def}}{=} [f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + (x_n)]\,\Delta x = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Let $J = \sup_{x\in[a,b]} |f'(x)|$. Then we can bound the error ($E_L = L_n - \int_a^b f(x)\,dx$ and $E_R = R_n - \int_a^b f(x)\,dx$) of the left- and right-endpoint approximations:

$$|E_L|\,, |E_R| \le \frac{J(b-a)^2}{2n}$$

Trapezoid Rule Divide $[a, b]$ into n equal pieces. Let $\Delta x = (b - a)/n$ be the width of the piece, and $x_i = a + i\Delta x$ be the ith division (so $x_0 = a$ and $x_n = b$). To each piece, associate the average of the left- and right-endpoint heights:

$$\begin{aligned}\int_a^b f(x)\,dx \approx T_n &\stackrel{\text{def}}{=} \left[\frac{1}{2}f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{1}{2}f(x_n)\right]\Delta x \\ &= \sum_{i=1}^{n} \frac{f(x_{i-1}) + f(x_i)}{2}\Delta x\end{aligned}$$

Let $K = \sup_{x\in[a,b]} |f''(x)|$. Then we can bound the error of the trapezoid approximation:

$$|E_T| \stackrel{\text{def}}{=} \left|T_n - \int_a^b f(x)\,dx\right| \le \frac{K(b-a)^3}{12n^2}$$

Midpoint Rule Divide $[a, b]$ into n equal pieces. Let $\Delta x = (b - a)/n$ be the width of the piece, and $\bar{x}_i = a + \left(i + \frac{1}{2}\right)\Delta x = (x_{i-1} + x_i)/2$ be the midpoint of each piece. To each piece, associate the height at this midpoint:

$$\int_a^b f(x)\,dx \approx M_n \stackrel{\text{def}}{=} [f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_{n-1}) + (\bar{x}_n)]\,\Delta x = \sum_{i=1}^{n} f(\bar{x}_i)\,\Delta x$$

Let $K = \sup_{x\in[a,b]} |f''(x)|$. Then we can bound the error of the midpoint approximation:

$$|E_M| \stackrel{\text{def}}{=} \left| M_n - \int_a^b f(x)\,dx \right| \leq \frac{K(b-a)^3}{24n^2}$$

Simpson's Rule (I'm going to use different numbering from the textbook.) Divide $[a, b]$ into n equal pieces. Let $\Delta x = (b - a)/n$ be the width of the piece, $x_i = a + i\Delta x$ be the ith division (so $x_0 = a$ and $x_n = b$), and $\bar{x}_i = a + \left(i + \frac{1}{2}\right)\Delta x = (x_{i-1} + x_i)/2$ be the midpoint of each piece. On each piece, approximate the function by a parabola:

$$\begin{aligned}\int_a^b f(x)\,dx \approx S_n &\stackrel{\text{def}}{=} \left[\frac{1}{6}f(x_0) + \frac{2}{3}f(\bar{x}_1) + \frac{1}{3}f(x_1) + \frac{2}{3}f(\bar{x}_2) + \frac{1}{3}f(x_2) + \ldots\right. \\ &\qquad \left. + \frac{2}{3}f(\bar{x}_{n-1}) + \frac{1}{3}f(x_{n-1}) + \frac{2}{3}f(\bar{x}_n) + \frac{1}{6}f(x_n)\right]\Delta x \\ &= \sum_{i=1}^{n} \frac{f(x_{i-1}) + 4f(\bar{x}_i) + f(x_i)}{6}\Delta x\end{aligned}$$

Let $L = \sup_{x\in[a,b]} \left|f^{(4)}(x)\right|$. Then we can bound the error of the midpoint approximation:

$$|E_S| \stackrel{\text{def}}{=} \left| S_n - \int_a^b f(x)\,dx \right| \leq \frac{L(b-a)^5}{180n^4}$$

In all of these estimates, the error is no more than something depending on the maximum value of some derivative of the function. Often, actually finding this maximum value is very difficult. Instead, one can always find some J, K, or L larger than the maximum value; this only increases your estimate of the error, which may still be quite small. The *triangle law* is very helpful: $|A + B| \leq |A| + |B|$.

1. Let $f(x)$ be a positive increasing function with negative second derivative on $[a, b]$. Place the following five numbers in increasing order: L_n, R_n, T_n, M_n, and $\int_a^b f(x)dx$.

2. If you evaluate each of the following integrals using each of the Midpoint, Trapezoid, and Simpson Rules with 5 subintervals, what is your expected error? How many pieces do you need to take to get an error less than 0.00001?

   (a) $\displaystyle\int_0^{\frac{1}{2}} \sin^2 x\,dx$ (b) $\displaystyle\int_0^3 \frac{dt}{1 + t + t^4}$ (c) $\displaystyle\int_0^4 \sqrt{1 + \sqrt{x}}\,dx$

3. Show that $(T_n + M_n)/2 = T_{2n}$. Show that $(T_n + 2M_n)/3 = S_n$.

4. Show that Simpson's rule calculates the area under a cubic curve exactly. What are the highest degree polynomials the rest of the approximation rules calculate exactly?