Math 1B Handout: Estimating Integrals by Sums

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To approximate $\int_a^b f(x)\,dx$, let $\Delta x = (b-a)/n$, $x_i = a + i\Delta x$, and $\bar{x}_i = (x_{i-1} + x_i)/2$. Then define the following approximations (we use different number from Stewart for Simpson's Rule S_n):

$$\begin{aligned}
L_n &= (f(x_0) + \cdots + f(x_{n-1}))\,\Delta x \\
R_n &= (f(x_1) + \cdots + f(x_n))\,\Delta x \\
T_n &= (f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n))\,\frac{\Delta x}{2} \\
M_n &= (f(\bar{x}_1) + \cdots + f(\bar{x}_n))\,\Delta x \\
S_n &= (f(x_0) + 4f(\bar{x}_1) + 2f(x_1) + 4f(\bar{x}_2) + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(\bar{x}_n) + f(x_n))\,\frac{\Delta x}{6}
\end{aligned}$$

These have the following errors:

$$\begin{aligned}
|E_L|\,, |E_R| &\le \sup_{x\in[a,b]} \left|f'(x)\right| \frac{(b-a)^2}{2n} \\
|E_T|\,, 2\,|E_M| &\le \sup_{x\in[a,b]} \left|f''(x)\right| \frac{(b-a)^3}{12n^2} \\
|E_S| &\le \sup_{x\in[a,b]} \left|f^{(4)}(x)\right| \frac{(b-a)^5}{180n^4}
\end{aligned}$$

1. Let $f(x)$ be a positive increasing function with negative second derivative on $[a,b]$. Place the following five numbers in increasing order: L_n, R_n, T_n, M_n, and $\int_a^b f(x)dx$.

2. If you evaluate each of the following integrals using each of the Midpoint, Trapezoid, and Simpson Rules with 5 subintervals, what is your expected error? How many subintervals do you need to take to get an error less than 0.00001?

   (a) $\displaystyle\int_0^{\frac{1}{2}} \sin^2 x\,dx$ (b) $\displaystyle\int_0^3 \frac{dt}{1+t+t^4}$ (c) $\displaystyle\int_0^4 \sqrt{1+\sqrt{x}}\,dx$

3. Show that $(T_n + M_n)/2 = T_{2n}$. Show that $(T_n + 2M_n)/3 = S_n$.

4. By explicit calculation, show that Simpson's rule calculates the area under a cubic curve exactly. What are the highest degree polynomials the rest of the approximation rules calculate exactly?

5. By explicit calculation, show that the errors for L_n and R_n are exact when $f(x)$ is a linear function, and that the errors for T_n and M_n are exact when $f(x)$ is a quadratic function.

6. Every polynomial can be exactly integrated. Do this! Evaluate:

$$\int_0^1 \left(a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0\right) dx$$

But not every rational function can be exactly integrated: $\int_1^x \frac{dt}{t} = \ln(x)$ and $\int_0^x \frac{dt}{1+x^2} = \arctan(x)$, neither of which are rational functions.

However, even after we've invented all the trigonometric and exponential functions, and accurately computed their values into a table, so that we can easily compose them, there are still functions that are hard to integrate. For instance, it is a theorem that e^{x^2} does not have an integral that is expressible in elementary functions, and yet it turns up all the time, especially in statistics. So let's invent a new function, called the "Gaussian":

$$G(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2} dt$$

(a) In terms of $G(x)$, evaluate

$$\int_a^1 \sqrt{\ln \frac{1}{y}}\, dy$$

where a is some given number with $0 < a < 1$.

(b) Pick your favorite approximation technique. How many numbers would you need to add together to compute $G(1)$ to within a tenth-of-a-percent error (0.001)?