Math 1B Handout: Power Series

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If a power series $\sum_{n=0}^{\infty} c_n x^n$ has radius of convergence $R > 0$, then the integral $C + \sum_{n=0}^{\infty} c_n x^{n+1}/(n+1) = C + \sum_{n=1}^{\infty} c_{n-1} x^n/n$ and the derivative $\sum_{n=0}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)c_{n+1}x^n$ also have radius of convergence equal to R. Endpoints are subtler: if $\sum c_n x^n$ converges at an endpoint, then so does its integral, and if it diverges at an endpoint, then so does its derivative.

Remember that we have the following power series representation:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots = \sum_{n=0}^{\infty} x^n$$

1. (a) Write down the power series expansion of $f(x) = 1/(1+x)$.
   (b) Integrate both sides of your equation from part (a) to get a power series expansion of $\ln(1+x)$. What is the radius of convergence? Does this function converge at the endpoints? Use this to write down a series for $\ln(2)$.
   (c) What is the integral of $\ln(1+x)$ in terms of functions (not power series)? Now, use the power series from (b) to get a power series for this integral. How does this compare to the power series you get if you integrate the series in (b)?

2. Integrate the power series for $1/(1+x^2)$. What's the radius of convergence of your series? Does it converge at the endpoints? What equation do you get when you substitute $x = 1$? What about when $x = 1/\sqrt{3}$?

3. Find a power series representation for the function and determine the interval of convergence:
   (a) $\dfrac{3}{1-x^4}$
   (b) $\dfrac{1}{1+9x^2}$
   (c) $\dfrac{x}{4x+1}$
   (d) $\dfrac{x^2}{a^3-x^3}$

4. By differentiating, find a power series representation for the function:

(a) $\dfrac{1}{(1-x)^2}$

(b) $\dfrac{1}{(1-x)^3}$

(c) $\dfrac{1}{(1-x)^n}$

(d) $\dfrac{1}{(1+x^2)^2}$

(e) $\dfrac{1}{(1+x^2)^3}$

(f) $\dfrac{1}{(1+x^2)^n}$

5. What is the lower series of $\ln(1-x)$? What about $x\ln(1-x)$? Convince yourself that any integral of $x\ln(1-x)$ is a polynomial times $\ln(1-x)$. What about if $\ln(1-x)$ is replaced by $\arctan(x)$?

6. Show that
$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$
is a solution to the differential equation $f'' + f = 0$. What is the function?

7. Use partial fractions to express the function
$$\frac{5x+1}{x^2-3x+2}$$
as a sum of power series. What is the interval of convergence?

8. (a) What power series do you get if you differentiate the power series for $1/(1-x)$?
   (b) What power series do you get if you differentiate again?
   (c) How would you write $\sum_{n=0}^{\infty} n^2 x^n$ as a function?
   (d) If $p(x)$ is any polynomial, use the ratio test to determine the radius of convergence of $\sum_{n=0}^{\infty} p(n)x^n$. Does this converge on the boundary?
   (e) Come up with a method that you could use to write $\sum_{n=0}^{\infty} p(n)x^n$ as a function for any given polynomial $p(n)$. For example, what is $\sum_{n=0}^{\infty}(3n^2-4n+1)x^n$?