Math 1B Handout: Taylor Series

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If $f(x)$ can be represented by a power series $f(x) = \sum_{n=0}^{\infty} c_n x^n$, then f is ∞-times differentiable, and the coefficients are given by

$$\boxed{c_n = \frac{f^{(n)}(0)}{n!}}$$

These are called the *Taylor coefficients for f at* 0, and the series $\sum_{n=0}^{\infty} f^{(n)}(0)\, x^n/n!$ is called the *Taylor series for f centered at* 0, or the *Maclaurin series for f*. By translating, we can also have *Taylor series centered at a*: $\sum_{n=0}^{\infty} f(n)(a)\,(x-a)^n/n!$.

Some infinitely-differentiable functions cannot be represented by power series. But: the Taylor series $\sum_{n=0}^{\infty} f^{(n)}(0)\, x^n/n!$ converges to $f(x)$ for $|x| < R$ if f satisfies that

$$\lim_{n\to\infty} \left(\frac{R^n}{n!} \sup_{|x|<R} \left| f^{(n)}(x) \right| \right) = 0$$

Most functions that are actually used — e.g. rational functions, exponential and trigonometric functions and their inverses, etc. — are equal to their Taylor series inside the radius of convergence.

1. Find the Taylor series expansion centered at 0 for $1/(1-x)$, by computing derivatives. Is this what you expect?

2. Write out the Taylor series expansion centered at 0 of the following functions. What is the radius of convergence?

   (a) $\sin 2x$
   (b) xe^x
   (c) $\cosh x$
   (d) $f(x) = 1/(2-x)$
   (e) $f(x) = \sin(2x - \pi/4)$

3. What is the Taylor series expansion of e^{-1/x^2} centered at 0? What is the radius of convergence of this series? Why should this make you troubled?

4. By using the Taylor series expansion for $\sin(x)$ only up to the cubic term, approximate the non-zero solution for

$$x^2 = \sin(x)$$

5. Prove that

$$\int_0^1 \frac{1+x^{30}}{1+x^{60}} dx = 1 + \frac{c}{31}$$

where $0 < c < 1$