Math 1B Handout: More Taylor Series

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Tuesday, 5 August 2008

Functions that equal their Taylor series expansion inside the interval of convergence are called *analytic*. Here are a few analytic functions and their Taylor series:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{1}$$

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \tag{2}$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \tag{3}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \tag{4}$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} \tag{5}$$

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1} \tag{6}$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n \tag{7}$$

The last equation is the *binomial theorem*. The symbol $\binom{k}{n}$ is defined as $(k)(k-1)\dots(k-n+1)/n!$, for any number k and integer n. (The numerator is a product of k numbers, just like the denominator.)

1. Write out the Taylor series expansion centered at 0 of the following functions. What is the radius of convergence?

   (a) $(3+x)^{1/3}$

   (b) $\ln(5+x)$

   (c) $\arctan(6x)$

(d) $\sin 2x$

(e) xe^x

(f) $\cosh x$

(g) $f(x) = 1/(2-x)$

(h) $f(x) = \sin(2x - \pi/4)$

2. What is the Taylor series expansion of e^{-1/x^2} centered at 0? What is the radius of convergence of this series? Why should this make you troubled?

3. By using the Taylor series expansion for $\sin(x)$ only up to the cubic term, approximate the non-zero solution for
$$x^2 = \sin(x)$$

4. Prove that
$$\int_0^1 \frac{1+x^{30}}{1+x^{60}} dx = 1 + \frac{c}{31}$$
where $0 < c < 1$