Math 1B Handout 7
Improper Integrals; Arc Length

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Improper Integrals

There was not a lot of time yesterday for improper integrals. Here are some of the problems again:

1. Is $\int_0^\pi \tan x\,dx$ well-defined (i.e. does the integral converge)? If so, to what?

2. Use the comparison test to decide if the following integrals converge:

$$\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx \qquad \int_1^\infty \frac{2+e^{-x}}{x}\,dx \qquad \int_0^{\pi/2} \frac{dx}{x\sin x} \qquad \int_1^\infty \frac{\arctan x}{x^2}\,dx$$

3. Evaluate the integral or show that it is divergent:

$$\int_0^1 \frac{t^2+1}{t^2-1}\,dt \qquad \int_2^6 \frac{y}{\sqrt{y-2}} \qquad \int_0^1 \frac{dx}{2-3x} \qquad \int_{-1}^1 \frac{x+1}{\sqrt[3]{x^4}}\,dx$$

4. In terms of b and c, evaluate

$$\int_{-\infty}^\infty \frac{dx}{x^2+bx+c}.$$

What conditions do you need to place on b and c to assure that this integral converges?

5. Find the value of the constant C for which the integral
$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2} \right) dx$$
converges. Evaluate the integral for this value of C.

6. Show that $\int_0^1 \ln x \, dx$ converges, and find the limit. More generally, use integration by parts to find $\int_0^1 (\ln x)^n \, dx$ for any nonnegative integer n.

7. Let n be a nonnegative integer. Using integration by parts, find $\int_0^\infty x^n \, e^{-x} \, dx$.

8. The "Laplace Transform" of a function $f(t)$ is the function of s given by
$$\mathcal{L}[f](s) = \int_0^\infty f(t) \, e^{-st} \, dt$$
if this integral converges. Find the Laplace transforms $\mathcal{L}[1](s)$, $\mathcal{L}[t](s)$, $\mathcal{L}[e^t](s)$, and $\mathcal{L}[t^n](s)$. (The last one requires your answer to the previous question.) What are the domains of these functions (for what s values do the integrals converge)?

Arc Length

The length of the curve $\gamma = \{y = f(x) : a \leq x \leq b\}$ is given by the formula
$$\ell(\gamma) = \int_a^b \sqrt{1 + (f'(x))^2} \, dx$$
For a curve of the form $x = g(y)$, we also have the arclength $\int \sqrt{1 + (g'(y))} dy$. In general, the arc length is $\int ds$, where $ds = \sqrt{dx^2 + dy^2}$.

1. Find the lengths of the curves:
$$y = \frac{x^2}{2} - \frac{\ln x}{4}, \qquad 2 \leq x \leq 4$$
$$y^2 = 4x, \qquad 0 \leq y \leq 2$$

2. Using the arc length formula, prove the formula for the circumference of a circle.

3. Set up, but do not evaluate, an integral for the perimeter of the ellipse $\{x^2/a^2 + y^2/b^2 = 1\}$. (Evaluating this integral is notoriously hard; integrals of this form are called, not surprisingly, "elliptical integrals.")

4. (a) Sketch the curve $\{y^3 = x^2\}$.
   (b) Set up two integrals, one in terms of x and one in terms of y, for the arc length of the above curve from $(0, 0)$ to $(1, 1)$. One of your integrals should be an improper integral. Evaluate each of them.
   (c) Find the length of the arc of this curve from $(-1, 1)$ to $(8, 4)$.