Math 1B Handout: $ay'' + by' + cy = 0$

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Second-order homogeneous linear DiffEQs with constant coefficients

Consider the differential equation

$$ay'' + by' + cy = 0$$

where a, b, and c are real numbers. Then $y = e^{rt}$ is a solution if and only if $ar^2 + br + c = 0$. If this equation has two distinct real roots, then the corresponding exponential functions are linearly independent, so their linear combinations exhaust all the solutions to the differential equation.

If $ar^2 + br + c = 0$ has only one root r, then e^{rt} is a solution to $ay'' + by' + cy = 0$, but so is te^{rt}.

If $ar^2 + br + c = 0$ has no real roots, then it must have two distinct complex roots $r = \alpha \pm i\beta$. By Euler's formula, $e^{(\alpha \pm i\beta)t} = e^{\alpha t} (\cos \beta t \pm i \sin \beta t)$, and so two linearly independent real solutions to the differential equation are $e^{\alpha t} \cos \beta t$ and $e^{\alpha t} \sin \beta t$.

1. Solve the following differential equations:
   (a) $y'' - 4y' + 8y = 0$
   (b) $2y'' - y' - y = 0$
   (c) $3y'' = 5y'$
   (d) $16y'' + 24y' + 9y = 0$
   (e) $9y'' + 4y = 0$

2. What if $a = 0$?
   (a) What are the solutions to $by' + cy = 0$? What are the roots to $br + c = 0$?
   (b) If a is not zero, then there are two solutions to $ar^2 + br + c = 0$. Evaluate the limits as $a \to 0$, holding b and c constant, of these solutions. Hint: $\sqrt{1 + x} \approx 1 + x/2$ when x is small.

3. (a) A frictionless spring is described by a second-order differential equation: the force (mass times acceleration) is proportional to the displacement of the spring.

If the mass is m and the "spring constant" (constant of proportionality) is k, write and solve a differential equation to find the most general equation for the position of the spring. (You need to know whether k is positive or negative: draw a picture to figure out which direction the force should push.)

(b) Often engineers place springs in viscous fluids in order to dampen the movement of the spring; if the fluid is jostled enough so that the internal flow is consistently turbulent, then the damping force will be proportional to the velocity (let's say with proportionality constant c). If $c^2 > 4mk$, what is the behavior of the spring?

(c) If $c^2 < 4mk$, then the solution to the differential equation is

$$\text{position} = e^{-ct/2m}\left(A\cos(t\omega) + B\sin(t\omega)\right)$$

By plugging into your differential equation, find the frequency ω.

Sketch the graph of this solution. Car shock absorbers are dampened springs; if your shock absorber has $c^2 > 4mk$, what would it feel like to go over a bump?

4. If the characteristic equation has zero real roots, then the solution in general is of the form

$$e^{\alpha x}\left(A\cos(\beta x) + B\sin(\beta x)\right)$$

If there are two real roots, we can use the same equation, but with the hyperbolic functions rather than the trigonometric ones. For what values of α and β is

$$y(x) = e^{\alpha x}\left(A\cosh(\beta x) + B\sinh(\beta x)\right)$$

a two-parameter family of solutions to

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

if $b^2 > 4ac$?

5. Show that if r is a double root of some polynomial $p(x)$, then r is also a root of $p'(x)$. Hint: what does the fact that r is a double root tell you about the factorization of p?

6. Let r_1 and r_2 be the two roots of $ar^2 + br + c = 0$, where $b^2 > 4ac > 0$. Holding a and c constant, take the limit as $b \to \sqrt{4ac}$ of the solution

$$y = \frac{1}{r_1 - r_2}e^{r_1 t} + \frac{1}{r_2 - r_1}e^{r_2 t}$$

to the differential equation $ay'' + by' + cy = 0$. Hint: L'Hôpital.