Math 1B Handout: $ay'' + by' + cy = g(t)$

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Nonhomogeneous linear equations, undetermined coefficients

A differential equation of the form $ay'' + by' + cy = g(t)$ has a two-parameter family of solutions, but the solution space is not a linear space. Instead, if $z_1(t)$ and $z_2(t)$ are solutions to $ay'' + by' + cy = g(t)$, then $z_1(t) - z_2(t)$ is a solution to $ay'' + by' + cy = 0$. This is called the "complementary equation". Thus, to solve $ay'' + by' + cy = g(t)$, find the general solution $c_1y_1(t) + c_2y_2(t)$ to $ay'' + by' + cy = 0$ and find some solution y_p to $ay''+by'+cy = g(t)$; then the general solution to $ay''+by'+cy = g(t)$ is $y_p+c_1y_1(t)+c_2y_2(t)$.

Some functions — notably, polynomials, exponentials, and trigonometric functions — have the property that they have finitely many linearly independent derivatives. In this case, chances are that some combination of the derivatives will provide a particular solution to the general equation. By guessing that a particular solution is some such combination, and then solving for the unknown coefficients, one can usually find the desired particular solution. Of course, the general solution is given by a particular solution, plus the general solution to the complementary equation. The guesses one should make:

- If $g(t) = e^{kt}p(t)$, where p is a polynomial of degree n, guess $y_p(t) = e^{kt}q(t)$, where q is a degree-n polynomial with undetermined coefficients.
- If $g(t) = e^{kt}p(t)\cos mt$ or $e^{kt}p(t)\cos mt$, guess $y_p(t) = e^{kt}q(t)\cos mt + e^{kt}r(t)\sin mt$.
- If $g(t) = g_1(t)+g_2(t)$, it might be simpler to solve each equation $ay''+by'+cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$ separately, and add the answers.
- If any $y(t)$ of the form of the guess $y_p(t)$ is itself a solution to the complementary equation $ay'' + by' + cy = 0$, you may have to multiply those terms by t or t^2.

1. Find the general solution to the differential equation:
   (a) $y'' + 9y = e^{3t}$
   (b) $y'' + 6y' + 9y = 1 + t$
   (c) $y'' + 6y' + 9y = t^2e^{-3t}$
2. What trial solutions would you use for the differential equation $ay''+by'+cy = g(x)$, if you use the method of "undetermined coefficients"?

(a) $g(t) = \sinh(t)$
(b) $g(t) = \sin(t)\cos(2t)$
(c) $g(t) = \tan(t)\sin(2t)$

3. (a) If you know trial solutions for $ay'' + by' + cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$, what is a trial solution for $ay'' + by' + cy = g_1(t) + g_2(t)$?
   (b) What about for $ay'' + by' + cy = g_1(t)g_2(t)$?