Math 1B Handout: $ay'' + by' + cy = g(t)$

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Undetermined coefficients

- If $g(t) = e^{kt}p(t)$, where p is a polynomial of degree n, guess $y_p(t) = e^{kt}q(t)$, where q is a degree-n polynomial with undetermined coefficients.
- If $g(t) = e^{kt}p(t)\cos mt$ or $e^{kt}p(t)\cos mt$, guess $y_p(t) = e^{kt}q(t)\cos mt + e^{kt}r(t)\sin mt$.
- If $g(t) = g_1(t) + g_2(t)$, it might be simpler to solve each equation $ay'' + by' + cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$ separately, and add the answers.
- If any $y(t)$ of the form of the guess $y_p(t)$ is itself a solution to the complementary equation $ay'' + by' + cy = 0$, you may have to multiply those terms by t or t^2.

1. Find the general solution to the differential equation:
   (a) $y'' + 9y = e^{3t}$
   (b) $y'' + 6y' + 9y = 1 + t$
   (c) $y'' + 6y' + 9y = t^2 e^{-3t}$
2. What trial solutions would you use for the differential equation $ay'' + by' + cy = g(x)$, if you use the method of "undetermined coefficients"?
   (a) $g(t) = \sinh(t)$
   (b) $g(t) = \sin(t)\cos(2t)$
   (c) $g(t) = \tan(t)\sin(2t)$
3. (a) If you know trial solutions for $ay'' + by' + cy = g_1(t)$ and $ay'' + by' + cy = g_2(t)$, what is a trial solution for $ay'' + by' + cy = g_1(t) + g_2(t)$?
   (b) What about for $ay'' + by' + cy = g_1(t)g_2(t)$?

Variation of parameters

We're trying to solve $ay'' + by' + cy = g(t)$. Let $y_1(t)$ and $y_2(t)$ be linearly independent solutions to the complementary equation $ay'' + by' + cy = 0$. If c_1 and c_2 are any constants, then $y(t) = c_1 y_1(t) + c_2 y_2(t)$ satisfies $ay'' + by' + cy = 0$. If $u_1(t)$ and $u_2(t)$ are almost constants and $y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$, then we expect that $ay'' + by' + cy$ is almost zero. More generally, $y_1(t)$ and $y_2(t)$ provide a convenient basis for the differential equation, so we should look for a particular solution of the form $y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$.

$$
\begin{aligned}
y(t) &= u_1(t)\, y_1(t) + u_2(t)\, y_2(t) \\
y'(t) &= u_1'(t)\, y_1(t) + u_2(t)'\, y_2(t) + u_1(t)\, y_1'(t) + u_2(t)\, y_2'(t) \\
y''(t) &= u_1''(t)\, y_1(t) + u_2''(t)\, y_2(t) + 2u_1'(t)\, y_1'(t) + \\
&\quad + 2u_2'(t)\, y_2'(t) + u_1(t)\, y_1''(t) + u_2(t)\, y_2''(t) \\
ay''(t) + by'(t) + cy(t) &= u_1(t)\left[ay_1''(t) + by_1'(t) + cy_1(t)\right] + \\
&\quad + u_2(t)\left[ay_2''(t) + by_2'(t) + cy_2(t)\right] + \\
&\quad + u_1'(t)\left[2ay_1'(t) + by_1(t)\right] + u_2'(t)\left[2ay_2'(t) + by_2(t)\right] + \\
&\quad + a\left[u_1''(t)\, y_1(t) + u_2''(t)\, y_2(t)\right] \\
&= 0 + 0 + u_1'(t)\left[2ay_1'(t) + by_1(t)\right] + u_2'(t)\left[2ay_2'(t) + by_2(t)\right] + \\
&\quad + a\left[u_1''(t)\, y_1(t) + u_2''(t)\, y_2(t)\right] \\
&= b\left[u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t)\right] + a\left[u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t)\right]' + \\
&\quad + a\left[u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t)\right]
\end{aligned}
$$

If we can find $u_1(t)$ and $u_2(t)$ such that $u_1'(t)\, y_1(t) + u_2'(t)\, y_2(t) = 0$, then our equation would simplify to $a\,[u_1'(t)\, y_1'(t) + u_2'(t)\, y_2'(t)] = g(t)$. Thus, we win if we can find u_1 and u_2 satisfying both these equations. Solving the equations in u_1' and u_2' gives

$$
\begin{aligned}
u_1'(t) &= \frac{g(t)\, y_1(t)}{a\,(y_1(t)\, y_2'(t) - y_2(t)\, y_1'(t))} \\
u_2'(t) &= \frac{g(t)\, y_2(t)}{a\,(y_2(t)\, y_1'(t) - y_1(t)\, y_2'(t))}
\end{aligned}
$$

If y_1 and y_2 are linearly independent, then the denominator is non-zero. By integrating each of $u_1'(t)$ and $u_2'(t)$, we find the solution $y(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$ of the original equation $ay'' + by' + cy = g(t)$.

1. Let's say we're trying to solve $ay'' + by' + cy = g(t)$ by the method of variation of parameters, and let's say that $b^2 > 4ac$, so that the characteristic equation has two real solutions $y_i = e^{r_i t}$ ($i = 1$ or 2). We're looking for $y_p(t) = u_1(t)\, y_1(t) + u_2(t)\, y_2(t)$. Explicitly solve the system of linear equations you get for $u_1'(t)$ and $u_2'(t)$ (in terms of $g(t)$, a, and r_1 and r_2).

2. Find the general solution to the differential equation:

$$y'' + 4y' + 3y = \frac{1}{1 + e^{2t}}$$

3. Here's another way to solve a second-order linear equation with constant coefficients. I will sketch the method in general; try it with the specific equation $4y'' - 4y' + y = e^t$ (which you could also solve by "undetermined coefficients"):

   (a) We're trying to find solutions $y = y(t)$ to $ay'' + by' + cy = g(t)$. Begin by finding a non-zero solution (you just need one) $y_0(t)$ to the homogeneous equation $ay_0'' + by_0' + cy_0 = 0$.

   (b) Now say we have some solution y to the nonhomogeneous equation. Whatever it is, we can divide by $y_0(t)$ to write $y(t) = z(t)\, y_0(t)$. Plug $y = zy_0$ into the nonhomogeneous equation, and use the product rule to expand out the derivatives.

   (c) Use the fact that y_0 is a solution to the homogeneous equation to simplify the expression, and combine like terms in derivatives of z. Notice that there are no terms in z (just in z' and z'').

   (d) Let $w(t) = z'(t)$; then your equation becomes a *first-order linear* differential equation in w. Solve this equation for $w(t)$.

   (e) Integrate this solution to get $z(t)$, and multiply by $y_0(t)$ to get $y(t)$.