Math 1B Handout 8
Arc Length and Surface Area

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Arc Length

The length of the curve $\gamma = \{y = f(x) : a \leq x \leq b\}$ is given by the formula

$$\ell[\gamma] = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

For a curve of the form $x = g(y)$, we also have the arclength $\int \sqrt{1 + (g'(y))}dy$. In general, the arc length is $\int ds$, where $ds = \sqrt{dx^2 + dy^2}$.

1. Find the lengths of the curves:
$$\begin{aligned} y = \frac{x^2}{2} - \frac{\ln x}{4}, &\qquad 2 \leq x \leq 4 \\ y^2 = 4x, &\qquad 0 \leq y \leq 2 \end{aligned}$$

2. Using the arc length formula, prove the formula for the circumference of a circle.

3. Set up, but do not evaluate, an integral for the perimeter of the ellipse $\{x^2/a^2 + y^2/b^2 = 1\}$. (Evaluating this integral is notoriously hard; integrals of this form are called, not surprisingly, "elliptical integrals.")

4. (a) Sketch the curve $\{y^3 = x^2\}$.

(b) Set up two integrals, one in terms of x and one in terms of y, for the arc length of the above curve from $(0,0)$ to $(1,1)$. One of your integrals should be an improper integral. Evaluate each of them.

(c) Find the length of the arc of this curve from $(-1,1)$ to $(8,4)$.

Surface Area of a surface of revolution

When the curve $\gamma = \{y = f(x) : a \le x \le b\}$ is rotated around the x-axis, the surface area of the surface of revolution is

$$\mathcal{A}_x[\gamma] = 2\pi \int_a^b f(x)\,\sqrt{1+(f'(x))^2}\,dx$$

If the same curve is rotated around the y-axis, the surface area if

$$\mathcal{A}_y[\gamma] = 2\pi \int_a^b x\,\sqrt{1+(f'(x))^2}\,dx$$

1. Find the areas traced out by rotating the curve around the x-axis:

$$\begin{array}{rl} y = \cos 2x & \quad 0 \le x \le \pi/6 \\ y = \dfrac{x^3}{6} + \dfrac{1}{2x} & \quad \dfrac{1}{2} \le x \le 1 \\ y = \ln(x^2+1) & \quad 0 \le x \le 1 \\ y = e^{2x} & \quad 0 \le x \le 1 \end{array}$$

2. Find the surface area of the ellipsoid formed by rotating $\{x^2/a^2 + y^2/b^2 = 1\}$ (where $a > b$) around the x-axis.

3. Find the surface area of the torus formed by rotating $\{(x-3)^2 + y^2 = 1\}$ around the y-axis.

4. Consider a sphere of radius 1 centered at the origin, sliced by two planes $x = a$ and $x = b$, $b > a$. Find the surface area between the two slices.

5. Consider infinite region above the x-axis, to the right of $x = 1$, and below the curve $y = 1/x$. Rotate this region around the x-axis. Show that the volume of revolution has infinite surface area but finite volume: If you fill this "cone" with paint, it will only hold finitely much, but it takes infinitely much paint to coat the inside wall.

6. Let f be a positive differentiable function on $[0,1]$. In the notation above, show that $\mathcal{A}_x[f+1] = \mathcal{A}_x[f] + 2\pi\ell[f]$.