Math 1B Handout: Sequences

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Wednesday, 23 July 2008

A *sequence* is an infinite list of numbers. In this class, and in most of math, we generally begin at 0: the beginning of a sequence is called the "zeroth" term, and then there's the "first" term, then the "second", etc. But really the indexing — is the sequence $\{a_0, a_1, a_2, \ldots\}$ or $\{a_4, a_5, a_6, \ldots\}$ — is largely irrelevant. In any case, the formal definition is that "A *sequence* is a function from the non-negative integers to the real numbers."

A sequence might or might not *converge* to a number, meaning it gets closer and closer to that number. A definition: $\lim_{n\to\infty} a_n = L$ if $|L - a_n|$ is small for sufficiently large n, i.e. if for any given $\epsilon > 0$ there exists an N such that for all $n > N$, $\epsilon > |L - a_n|$.

Whether, and to what, a sequence converges does not depend on the "early" values (or indeed on any given term) of a sequence. If a_n and b_n each converge, then so do $a_n + b_n$ and $a_n b_n$, and the limits add and multiply correctly.

If a sequence is monotonic — strictly increasing or strictly decreasing — and bounded, then it definitely converges.

A sequence can also *converge to* $+\infty$ or to $-\infty$. a_n converges to $+\infty$ if the terms increase without bound, i.e. if for any given $\epsilon > 0$ there exists an N such that for all $n > N$, $\epsilon < a_n$. An unbounded monotonic sequence converges to infinity.

1. (a) Show that the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$ definitely converges. This takes two steps: show that each term is bigger than the previous, and show that no term is bigger than 3 (by induction: show that if a_n is less than 3, then so is a_{n+1}).

   (b) Find the limit of the sequence $\{\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \ldots\}$, by finding an equation the limit must satisfy.

2. Show that the sequence defined by $a_0 = 2$ and $a_{n+1} = 1/(3 - a_n)$ is positive and decreasing. Hence it must be convergent. Find the limit.

3. (a) Let's say a sequence s_n diverges to $+\infty$. What is the limit of $1/s_n$ as $n \to \infty$? Justify your answer.

   (b) Is there a sequence t_n such that $\lim_{n\to\infty} t_n = 0$, such that $t_n \neq 0$ for any n, and $1/t_n$ does not diverge to $+\infty$ nor to $-\infty$?

(c) If you know that $t_n > 0$ for every n and that $t_n \to 0$ as $n \to \infty$, then what can you say about $\lim_{n\to\infty} 1/t_n$?

4. Determine whether the following sequences converge or diverge:

(a) $\dfrac{n+1}{3n-1}$

(b) $\dfrac{\sqrt{n}}{1+\sqrt{n}}$

(c) $\dfrac{n}{1+\sqrt{n}}$

(d) $\cos(2/n)$

(e) $n \cos n\pi$

(f) $(-3)^n/n!$