Math 1B Handout: Infinite Series

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A *series* is an infinite sum of numbers. There are two sequences associated to each series. First of all, there's the sequence of terms in the sum (a.k.a. "summands"):

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots \rightsquigarrow 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$$

Second, there's the sequence of partial sums:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots \rightsquigarrow 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots$$

We normally write the former explicitly:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots = \sum_{n=0}^{\infty} \frac{1}{2^n}$$

On the other hand, a series *converges* exactly if the sequence of partial sums converges, and the limit of the sequence of partial sums is the "value" of the series. Standard notation: the sequence of summands is a_n, the sequence of partial sums is $s_n = \sum_{k=0}^{n} a_k$, and if $\{s_n\}$ converges, then $\sum_{n=0}^{\infty} a_n = \lim_{n\to\infty} s_n$.

Here are some facts about series:

- A series cannot converge unless the sequence of summands tends to 0. But the sequence of summands can go to 0 without the series converging.
- The *geometric series* $a + ar + ar^2 + ar^3 + \ldots = \sum_{n=0}^{\infty} ar^n$ converges if and only if $|r| < 1$. If it converges, then it converges to $a/(1-r)$. (A proof and generalization of this is in problem 3.)
- If $\sum a_n = A$ and $\sum b_n = B$, then $\sum(a_n + b_n) = A + B$. Let $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Then $\sum c_n = AB$. This is called the "Cauchy product" or "discrete convolution" of the two series.
- This one isn't so much a fact as a technique. If we can write each a_n as a difference $a_n = b_n - b_{n+1}$, then $s_n = b_0 - b_{n+1}$, so $\sum a_n$ converges if $b_n \to 0$. Partial fractions are a good way to find such decompositions of terms as differences.

1. What is the sum

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots ?$$

How about

$$\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\ldots ?$$

If k is some number strictly greater than 1, what is

$$\sum_{n=1}^{\infty}\left(\frac{1}{k}\right)^n = \frac{1}{k}+\frac{1}{k^2}+\frac{1}{k^3}+\frac{1}{k^4}+\ldots ?$$

2. Determine whether the following series are convergent or divergent. If convergent, find the sums:

(a) $\displaystyle\sum_{n=0}^{\infty}\frac{n+1}{2n-3}$

(b) $\displaystyle\sum_{n=0}^{\infty}\frac{n(n+2)}{(n+3)^2}$

(c) $\displaystyle\sum_{n=0}^{\infty}\frac{1+3^n}{2^n}$

(d) $\displaystyle\sum_{n=0}^{\infty}[(.8)^n-(.3)^n]$

(e) $\displaystyle\sum_{n=0}^{\infty}(\cos 1)^n$

(f) $\displaystyle\sum_{n=0}^{\infty}\frac{2}{n^2-1}$

(g) $\displaystyle\sum_{n=0}^{\infty}\frac{3}{n(n+3)}$

3. Remember how to evaluate the geometric series: if $|r|<1$, then we can evaluate $S=\sum_{n=0}^{\infty} ar^n = a+ar+ar^2+ar^3+\ldots$ by mutiplying by r and subtracting:

$$\begin{array}{rrcl} & S & = & a+ar+ar^2+ar^3+\ldots \\ -\ (& r\cdot S & = & \quad ar+ar^2+ar^3+\ldots\ ) \\ \hline & S-rS & = & a \\ & & & \\ \text{thus} & S & = & \dfrac{a}{1-r}. \end{array}$$

Use this method to compute the following sums:

(a) $1+\frac{2}{3}+\frac{3}{9}+\frac{4}{27}+\frac{5}{81}+\frac{6}{243}+\ldots$

(b) $1+\frac{4}{2}+\frac{9}{4}+\frac{16}{8}+\frac{25}{32}+\frac{36}{64}+\ldots$