Math 1B Handout: Integral and Comparison Tests

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/08Summer1B/

Friday, 25 July 2008

The Integral Test: Let $f(x)$ be a continuous function that is positive and decreasing, at least after some cut-off. Then $\int_0^\infty f(x)\,dx$ converges if and only if $\sum_0^\infty f(n)$ converges.

The Comparison Test: Let $\{a_n\}$ and $\{b_n\}$ be two sequences, with $0 \leq b_n \leq a_n$ for every n. Then $0 \leq \sum b_n \leq \sum a_n$, and in particular if $\sum a_n$ converges, then so does $\sum b_n$.

1. Determine whether the following series converge or diverge:
   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^2-1}{3n^4+1}$
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2n+3}$
   (c) $\displaystyle\sum_{n=0}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$
   (d) $\displaystyle\sum_{n=0}^{\infty} \sin\left(\frac{1}{n}\right)$
2. Show that if $\sum a_n$ is a convergent series with positive terms, then $\sum \ln(1+a_n)$ also converges. What about $\sum \sin(a_n)$? If $\sum a_n$ diverges, what can you say about $\sum \ln(1+a_n)$ and $\sum \sin a_n$?
3. Let $\sum a_n$ be a convergent series and $\{b_n\}$ be a convergent sequences. Show that $\sum a_n b_n$ converges.
4. (a) For what values of p does $\sum 1/n^p$ converge?
   (b) For what values of p does $\sum 1/(n\,(\ln n)^p)$ converge? You may assume that the series starts after $n=1$.
   (c) For what pairs of values (p_0, p_1) does
   $$\sum \frac{1}{n^{p_0}\,(\ln n)^{p_1}}$$
   converge? You may assume that the series starts after $n=1$.
   (d) For what $(k+1)$-tuples $(p_0, p_1, \ldots, p_k)$ does
   $$\sum \frac{1}{n^{p_0}\,(\ln n)^{p_1}\,(\ln\ln n)^{p_2}\,\ldots\,(\underbrace{\ln\ldots\ln}_{k} n)^{p_k}}$$

converge? You may assume that the series starts late enough so as never to have 0s in the denominator.

5. This exercise provides a proof that there are infinitely many prime numbers. A *prime number*, like 2, 3, 5, or 101, is a positive number with exactly two positive factors. This exercise uses the fact that any positive number can be written as the product of prime numbers in exactly one way.

   (a) "Expand out" the following product into a single series:

   $$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots\right)\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\dots\right)$$

   What is the rule that determines whether a fraction does or does not appear in the (expanded out) sum? What is the value of the total sum?

   (b) How about the triple product:

   $$\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\dots\right)\left(1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\dots\right)\left(1+\frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\dots\right)?$$

   (c) What if you were to do this for *every* prime number:

   $$\left(1+\frac{1}{2}+\dots\right)\left(1+\frac{1}{3}+\dots\right)\left(1+\frac{1}{5}+\dots\right)\left(1+\frac{1}{7}+\dots\right)\left(1+\frac{1}{11}+\dots\right)\dots$$

   (d) Use the integral test, or otherwise, to check if the total "expanded out" sum converges or diverges.

   (e) Each of the individual multiplicands has a finite value (what is it?). Thus, conclude that there must be infinitely many multiplicands, and hence infinitely many prime numbers.