Math 1B Handout 9
Complex Numbers

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Complex Numbers

All the rules for manipulating complex numbers come from the definition: $i^2 = -1$. But some techniques are worth learning. To each complex number we associate its "complex conjugate", found by substituting $-i$ for i everywhere. This is a useful number, since the product of a number and its conjugate is always a positive real number. Another useful trick is the Euler formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

1. Simplify the following expression:

$$\sqrt{\frac{5}{1-2i} + (2+i)(-1+i) + 2 - \sqrt{3}}$$

2. Find all solutions to the equation:

$$2x^2 - 2x + 1 = 0$$

3. The quadratic formula solves any equation of the form $ax^2 + bx + c = 0$ in complex numbers. In this problem, you'll derive a similar formula for cubic equations.

(a) By substituting $z = w - 2/w$, solve the equation

$$z^3 + 6z = 20$$

(b) By substituting $z = w - p/(3w)$, find a formula for the three solutions to the equation

$$z^3 + pz = q$$

(c) By substituting $x = z + 2$ and using the formula in part (b), solve the equation

$$z^3 - 6z^2 - 5z + 22 = 0$$

(d) By substituting $x = z - a/3$ and using the formula in part (b), find a formula for the three solutions to the equation

$$z^3 + az^2 + bz + c$$

4. Use Euler's formula to prove the following formulas for sine and cosine:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad\qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

5. Use Euler's formula to find “triple angle fomulas” expressing $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

6. (a) If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable x, then the derivative of u is defined component-wise: $u'(x) = f'(x) + ig'(x)$. Let a and b be real constants. Use Euler's formula to evaluate

$$\frac{d}{dx}\left[e^{(a+ib)x}\right]$$

(b) In the above set-up, the indefinite integral $\int u(x)\,dx$ of $u(x)$ is any antiderivative. Evaluate

$$\int e^{(1+i)x}\,dx$$

(c) By considering real and imaginary parts of the above integral, compute

$$\int e^x \cos x\,dx \qquad \text{and} \qquad \int e^x \sin x\,dx$$