Math 1B Handout: Alternating Series, Absolute Convergence, and the Ratio Test

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The Alternating Series Test: Let b_n be a positive decreasing sequence: $b_n \geq b_{n+1} \geq 0$ for every n. Then $\sum (-1)^n b_n$ converges.

Absolute versus Conditional Convergence: A series $\sum a_n$ is said to *converge absolutely* if the absolute-value series $\sum |a_n|$ converges. An absolutely convergent series converges. A series that converges but does not converge absolutely is *conditionally convergent*, since the convergence is conditioned on the signs of the terms.

The Ratio Test: If $\lim_{n\to\infty} |a_{n+1}/a_n|$ exists and is less than 1, then the series $\sum a_n$ converges absolutely. If the limit of ratios exists and is more than 1 (or is $+\infty$), then the series diverges. (If the limit equals 1 or fails to exist, then the ratio test is inconclusive.)

1. Determine whether the following series are absolutely convergent, conditionally convergent, or divergent:

   (a) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}}$

   (b) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\ln n}{n}$

   (c) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{n}{n^2+1}$

   (d) $\displaystyle\sum_{n=0}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$

   (e) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin 4n}{4^n}$

   (f) $\displaystyle\sum_{n=0}^{\infty} \frac{1+(-1)^n n + (1/2)^n n^2}{n^2}$

   (g) $\displaystyle\sum_{n=0}^{\infty} \frac{n}{3^n}$

   (h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

   (i) $\displaystyle\sum_{n=0}^{\infty} \frac{3^n}{n!}$

   (j) $\displaystyle\sum_{n=0}^{\infty} \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-3)\cdot(2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n-2)\cdot(2n)}$

2. For what values of p and r do the following series converge absolutely?

   (a) $\displaystyle\sum_{n=1}^{\infty} r^n\, n^p$ (b) $\displaystyle\sum_{n=1}^{\infty} r^n\, (n!)^p$

3. (a) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n$ diverges, but $\sum_{n=1}^{\infty}(a_n)^2$ converges.

   (b) Find a sequence $\{a_n\}$ so that $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty}(a_n)^2$ diverges.

4. Often we use integrals to test whether series converge or not. Sometimes it's useful to go the other way: use the Alternating Series theorem to prove that $\displaystyle\int_0^{\infty} \frac{\sin(x)}{x}\, dx$ converges. **Warning:** a priori, this series could diverge both at $x \to \infty$ and at $x = 0$ (since we can't divide by 0, and so $\frac{\sin(x)}{x}$ isn't well-defined at the end-point); why does the integral actually pose no problems at $x = 0$?

5. (a) Show that, for any polynomial $f(x) = f_d x^d + f_{d-1}x^{d-1} + \ldots + f_1 x + f_0$, the ratio test fails to show that $\sum_{n=1}^{\infty} f(n)$ diverges. What's a test that does work?

   (b) Show that, for any polynomial $f(x)$, the series $\sum_{n=1}^{\infty} r^n f(n)$ converges if $|r| < 1$.