Math 1B Handout: Alternating Series, Absolute Convergence, and the Ratio Test

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A power series $\sum c_n(x-a)^n$ has an *interval of convergence*, which are the numbers x such that the power series converges. Everything inside the interval of convergence is absolute; everything outside diverges. The boundaries you don't know: for each boundary point, decide if it's absolute, conditional, or divergent.) The interval is necessarily centered at a: it has a *radius of convergence*.

1. For what x do the following series (i) converge absolutely? (ii) converge conditionally? (iii) diverge?
   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{(x-1)^n}{n!}$
   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{nx^n}{3^n}$
   (c) $\displaystyle\sum_{n=1}^{\infty} \frac{n(x-a)^n}{b^n}$ where a and b are fixed (but unknown) real numbers, and $b > 0$.
   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{n!}$
   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n x^n}{(n!)^2}$
2. If $\sum c_n 4^n$ converges, does it follow that $\sum c_n(-2)^n$ converges? What about $\sum c_n(-4)^n$?
3. (a) What is the interval of convergence for
$$f(x) = \sum_{n=1}^{\infty} nx^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \ldots?$$
   (b) What is the value of $f(x)$? There are two ways to do this problem (so pick one):

i. Multiply $f(x)$ by x, and subtract $f(x)-xf(x)$. Do you recognize this power series? Evaluate $f(x)-xf(x)$ and use that to solve for $f(x)$.

ii. Integrate $F(x) = \int_0^x f(t)\,dt$ term-by-term. Do you recognize this power series? Evaluate $F(x)$ and differentiate to get $f(x) = F'(x)$.

4. Let k be a positive integer. What is the radius of convergence of

$$\sum_{n=0}^{\infty} \frac{(n!)^k}{(kn)!} x^n$$

5. Show that if $\lim_{n\to\infty} \sqrt[n]{|c_n|} = c$, where $c \neq 0$, then the radius of convergence of the power series $\sum c_n x^n$ is $R = 1/c$.

6. Suppose the series $\sum c_n x^n$ has radius of convergence 2 and the series $\sum d_n x^n$ has radius of convergence 3. What is the radius of convergence of the series $\sum (c_n + d_n) x^n$?

7. Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{2n}$?