Math 1B Handout 2
Trigonometry

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

More integration by parts

1. What is $\int e^x \cos x\, dx$? How about $\int e^{ax} \cos x\, dx$, where a is a constant? How about $\int x\, e^x \cos x\, dx$?

2. What's wrong with the following proof that $0 = 1$?

$$\ln x = \int \frac{1}{x}\, dx = \frac{1}{x}\, x - \int \frac{-1}{x^2}\, x\, dx = 1 + \int \frac{1}{x}\, dx = 1 + \ln x$$

3. Why can we forget to add an arbitrary constant during the intermediate steps when integrating by parts? Integrate $\int x^n e^x dx$ completely honestly: let $u = x^n$ and $dv = e^x dx$, but this time let $v = e^x + C$.

4. Let $n = 2k + 1$ be an odd integer. Calculate $\int_{x=0}^{\pi/2} \cos^n(x)\, dx$ in two different ways:

   (a) Using a reduction formula. What happens to the boundary terms (the uv in $\int u\, dv = uv - \int v\, du$)? Why does it matter that n is odd? (The formula is different for even n.)

(b) Using a u-substitution. (Hint: $\cos^2 x = 1 - \sin^2 x$.) For any given k, you could then expand out and evaluate the integral. For a general k, you can use integration by parts to get a reduction formula.

Trigonometry

As in the above problem 4(b), the most important rule to know for trigonometric integrals is the Pythagorean identity:

$$\boxed{\cos^2 x + \sin^2 x = 1}$$

This lets us translate between (squares of) cosines and (squares of) sines. This is helpful for finding u-substitutions, since $\cos' = -\sin$ and $\sin' = \cos$. For example, to integrate $\cos^7(x)$, we can break off a cos and write the rest in terms of sin, and then substitute: $\int \cos^7 x\,dx = \int (\cos^2 x)^3 \cos x\,dx = \int (1-\sin^2 x)^3 \cos x\,dx = \int (1-u^2)^3 (du) = \int (-u^6 + 3u^4 - 3u^2 + 1) du = u^7/7 - 3u^5/5 + u^3 - u + C = \sin^7 x/7 - 3\sin^5 x/5 + \sin^3 x - \sin x + C$. This trick turns the integral of $\cos^n x \sin^m x$ into the integral of a polynomial provided that n and m are non-negative and at least one of n and m are odd. (When they can be negative, but still at least one is odd, we get rational functions, which we will learn how to integrate in section 6.3.)

By dividing the Pythagorean identity by $\sin^2$ or $\cos^2$, we get two more versions of the rule: $\boxed{1 + \tan^2 x = \sec^2 x}$ and $\boxed{1 + \cot^2 x = \csc^2 x}$. Since $\tan' = \sec^2$ and $\sec' = \tan \sec$, we can integrate $\tan^n \sec^m$ if m is even or n is odd. (It's similar for cot and csc.)

Sometimes, though, these aren't enough. Then it's important the remember the double-angle formulas:

$$\boxed{\sin^2 x = \frac{1}{2}(1 - \cos(2x)) \qquad \cos^2 x = \frac{1}{2}(1 + \cos(2x)) \qquad \sin x \cos x = \frac{1}{2}\sin(2x)}$$

1. Calculate $\int_{x=0}^{3\pi/2} (\sin x + \cos x)^3\,dx$.

2. Let a be a number such that $0 < a < \pi/2$. Compute the volume obtained by rotating the region bounded by the curves

$$y = \tan x,\ y = 0,\ x = a$$

   about the x-axis.

3. Find the average value of $\sin^2 x$: $\frac{1}{2\pi}\int_0^{2\pi} \sin^2 x\,dx$. Find the average values of $\sin^4 x$ and $\sin^2 x \cos^2 x$.

4. If you haven't already, be sure to do problem 4 above. Also find a formula for $\int_0^{\pi/2} \cos^n(x)\,dx$ when n is even.