Math 1B Handout 3
Trigonometric Substitutions

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

More Trigonometry

Here are a few more formulas you might want.

$$\begin{aligned} 2\sin(A)\sin(B) &= \cos(A-B) - \cos(A+B) \\ 2\sin(A)\cos(B) &= \sin(A-B) + \sin(A+B) \\ 2\cos(A)\cos(B) &= cos(A-B) + \cos(A+B) \end{aligned}$$

1. Integrate: $\int \cos(3x)\sin(2x)\,dx$
2. Integrate: $\int \cos(2x+1)\cos(4x-2)\sin(x)\,dx$

Trigonometric Substitutions

Many mathematical formulae involve the square roots of sums or differences of squares. Any integral involving, say, $\sqrt{a^2 \pm x^2}$, should make you think of the Pythagorean theorem, and hence trigonometry. The following versions of the Pythagorean formula are especially useful:

Pythagorean identity	Suggested substitution	Part of integral	$dx =$
$1-\sin^2\theta = \cos^2\theta$	$x = a\sin\theta$	$\sqrt{a^2-x^2} = a\cos\theta$	$dx = a\cos\theta$
$-1+\sec^2\theta = \tan^2\theta$	$x = a\sec\theta$	$\sqrt{x^2-a^2} = a\tan\theta$	$dx = a\tan\theta\sec\theta$
$1+\tan^2\theta = \sec^2\theta$	$x = a\tan\theta$	$\sqrt{a^2+x^2} = a\sec\theta$	$dx = a\sec^2\theta$

We always take θ in the range $0 \le \theta \le \pi/2$, so that all trig functions are positive. Feel free to change the bounds of integration, but attend to the signs of all terms and the domain in θ. The last integral is also particularly useful when the integral includes $1/(a^2 + x^2)$, even without a square root.

1. Try these integrals:

(a)	$\int_0^2 x^3\sqrt{x^2+4}\,dx$	(b)	$\int x^{-4}\sqrt{x^2-a^2}\,dx$
(c)	$\int t^5(t^2+2)^{-1/2}\,dx$	(d)	$\int_0^1 x\sqrt{x^2+4}\,dx$
(e)	$\int (u\sqrt{5-u^2})^{-1}\,du$	(f)	$\int_0^1 \sqrt{1+x^2}\,dx$
(g)	$\int (x^2\sqrt{16x^2-9})^{-1}\,dx$	(h)	$\int t(25-t^2)^{-1}\,dt$
(i)	$\int_0^{\pi/2} \cos t\,(1+\sin^2 t)^{-1/2}\,dt$	(j)	$\int x^n(1+x^2)^{-1}\,dx$

2. (a) What is the domain of the function $\sqrt{x^2+2x+2}$? Find numbers r and a such that $x^2+2x+2 = (x-r)^2 \pm a^2$.

   (b) Evaluate the following integrals: $\int \sqrt{x^2+2x+2}\,dx$. $\int (x^2+2x+2)^{-1/2}\,dx$. $\int (x^2+2x+2)^{-2}\,dx$.

3. (a) What is the domain of the function $\sqrt{x^2+4x-5}$? Find numbers r and a such that $x^2+4x-5 = (x-r)^2 \pm a^2$.

   (b) Evaluate the following integrals: $\int \sqrt{x^2+4x-5}\,dx$. $\int (x^2+4x-5)^{-1/2}\,dx$.

4. Find the average value of $f(x) = \sqrt{x^2-1}/x$ for $1 \le x \le 7$.

5. Draw the ellipses and find their areas:

   (a) $25x^2 + 9y^2 - 100x + 18y - 116 = 0$

   (b) $13x^2 + 13y^2 + 10xy = 25$

6. Imagine taking a solid sphere of radius 1, and slicing it by a plane slice a distance $a < 1$ from the center. What are the volumes of the two pieces?

7. You're standing on a pier. There's a boat in the water at distance L from you, connected to a rope of length L, and you're holding the other end (the rope is completely taut). Imagine that you start to walk along the pier, pulling the rope; sketch the path the boat follows.

   In fact, the boat will follow a path called a *tractrix*; it's defined by the property that the rope is always tangent to the path of the boat. To find an equation for the path as a function $y = y(x)$, solve the following integral:

$$\int \frac{dy}{y} = \int \frac{-\sqrt{L^2-x^2}}{x}dx$$