Math 1B Handout 4
Partial Fraction Decomposition

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/08Summer1B/

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Partial Fractions

When we add fractions, we go through all sorts of steps to find a common denominator, because for some purposes it's best to have an expressing written as a single fraction rather than as multiple fractions:

$$\frac{1}{2x+1} + \frac{x}{x^2+1} = \frac{x^2+x+1}{2x^3+x^2+2x+1}$$

But for integration, complicated fractions like $(x^2+x+1)/(2x^3+x^2+2x+1)$ are less than useful, whereas the simpler fractions are perfectly tractable:

$$\int \frac{x^2+x+1}{2x^3+x^2+2x+1}\,dx = \int \left(\frac{1}{2x+1} + \frac{x}{x^2+1}\right) dx = \frac{1}{2}\ln(2x+1) + \frac{1}{2}\ln(x^2+1) + C$$

(We compute the second integral by substituting $u = x^2+1$.)

In fact, every rational function (the ratio of two polynomials) can be decomposed uniquely into a sum of (a polynomial plus) simple fractions. A "simple fraction" is one where the denominator is a power of a linear or irreducible-quadratic polynomial, and the numerator is of lower degree than the root of the denominator.

The zeroth step is to perform long division so that your fraction is written as a "mixed number": a polynomial plus a fraction in which the numerator has lower degree than the denominator. The first step is to factor the denominator completely: $2x^3 + x^2 + 2x + 1 = (x^2 + 1)(2x + 1)$. Over the real numbers, every polynomial has a unique factorization into linear and irreducible-quadratic parts, but finding such a factorization in general can be very hard.

Then each factor in the denominator corresponds to a potential term in the decomposition, with repeated factors counting multiply. Write the most general decomposition given the factorization, and then solve for the unknown coefficients.

1. Decompose the following fractions into sums of simple fractions:
$$\frac{2x}{(x+3)(3x+1)} \qquad \frac{x-1}{x^3+x} \qquad \frac{2x+1}{(x+1)^3(x^2+4)^2}$$

2. Compute the following integrals. Remember that $\int dx/(x^2+1) = \arctan x$ and that $\int 2x\,dx/(x^2+1) = \ln(x^2+1)$. If necessary, don't forget to complete the square in the denominator:
$$\int \frac{r^2}{r+4}\,dr \qquad \int \frac{x-1}{x^2+3x+2}\,dx \qquad \int \frac{x^2}{(x-3)(x+2)^2}\,dx \qquad \int \frac{x^2-2x-1}{(x-1)^2(x^2+1)}\,dx$$

3. Solve $\int x/(x^2-1)\,dx$ (a) with a u-substitution, (b) with a trig substitution, (c) with a partial-fractions decomposition.

4. (a) Compute $\int \tan^2\theta \sec\theta\,d\theta$ by first performing a u-substitution and then decomposing into partial fractions.
   (b) Compute $\int (x^2+4)^{3/2}\,dx$ by first using a trig sub, then a u-sub, then partial fractions.

5. Factor x^4+1 as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4+1)\,dx$.

6. Any rational expression in sines and cosines can be made into a rational function via the *Weierstrass substitution*: $u = \tan(\theta/2)$.
   (a) Using trig identities, find general formulae for $\sin\theta$, $\cos\theta$, and $d\theta$ in terms of u.
   (b) Compute $\displaystyle\int \frac{d\theta}{3\cos\theta + 4\sin\theta}$.

7. If f is a quadratic function such that $f(0) = 1$ and $\displaystyle\int \frac{f(x)}{x^2(x+1)^3}\,dx$ is a rational function (no ln and arctan terms), find the value of $f'(0)$.

8. **This problem is only for those who know complex numbers.** Normally we integrate $\int dx/(x^2+1) = \arctan(x)$ with a trig substitution. Factor the denominator with complex numbers, decompose it as partial fractions, and integrate: what does this say about the relationship between arctan and ln?