Math 1B Handout 5
Integration

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Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Partial Fractions

There wasn't a lot of time yesterday to practice with partial fractions. Here are some of them again, and a few more.

1. Let's find the partial fraction decomposition of, say,

$$\frac{x+1}{x^3+x^2-2x}$$

   (a) Factor the denominator to find a, b, and c:

$$\frac{x+1}{x^3+x^2-2x} = \frac{x+1}{(x-a)(x-b)(x-c)}$$

   (b) We want to write

$$\frac{x+1}{x^3+x^2-2x} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

   We could put everything over a common denominator (i.e. multiply by $(x^3 + x^2 - 2x)$ and compare like terms. Instead, multiply both sides by just $(x-a)$, and then plug in $x = a$ into both sides of the equation. What does this tell you about the values of A, B, and C?

(c) Use this method to find A, B, and C.

(d) How would you do this for other examples? What about

$$\frac{x+1}{x^3-2x^2+x}? \qquad \frac{x+1}{x^3-x^2+x}?$$

2. Solve $\int x/(x^2-1)\,dx$ (a) with a u-substitution, (b) with a trig substitution, (c) with a partial-fractions decomposition.

3. (a) Compute $\int \tan^2\theta \sec\theta\, d\theta$ by first performing a u-substitution and then decomposing into partial fractions.

   (b) Compute $\int (x^2+4)^{3/2}\,dx$ by first using a trig sub, then a u-sub, then partial fractions.

4. Factor x^4+1 as a difference of squares by first adding and subtracting the same quantity. Use this factorization to evaluate $\int 1/(x^4+1)\,dx$.

5. Any rational expression in sines and cosines can be made into a rational function via the *Weierstrass substitution*: $u = \tan(\theta/2)$.

   (a) Using trig identities, find general formulae for $\sin\theta$, $\cos\theta$, and $d\theta$ in terms of u.

   (b) Compute $\displaystyle\int \frac{d\theta}{3\cos\theta + 4\sin\theta}$.

6. If f is a quadratic function such that $f(0) = 1$ and $\displaystyle\int \frac{f(x)}{x^2(x+1)^3}\,dx$ is a rational function (no ln and arctan terms), find the value of $f'(0)$.

7. **This problem is only for those who know complex numbers.** Normally we integrate $\int dx/(x^2+1) = \arctan(x)$ with a trig substitution. Factor the denominator with complex numbers, decompose it as partial fractions, and integrate: what does this say about the relationship between arctan and ln?

8. When a marble (with mass m, say) falls through a viscous liquid like honey, a constant downward force (gravity minus buoyancy $= \tilde{g} = mg - b$) acts on it, and friction impedes its motion with a force proportional to the square of the marble's velocity (say αv^2). Then the marble's velocity is given by the equation

$$\int dt = \int \frac{m\,dv}{\tilde{g} - \alpha v^2}$$

   To simplify the problem, let's let $m = \tilde{g} = \alpha = 1$ (or, if you want to, do the problem with all the unknown constants). Assume that the marble starts at rest at time $t = 0$. Solve this integral to find v as a function of t; don't forget to use the fact that $v(t = 0) = 0$ to figure out the constant of integration.

   For an extra challenge, find the distance the marble has fallen as a function of t.