Math 1B Handout 6
Improper Integrals

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/08Summer1B/

Monday, 30 June 2008

Please introduce yourselves to each other, and put your names at the top of a piece of blackboard. Take turns being the scribe: each of you should have a chance to write on the chalkboard for at least one of the exercises.

These groupwork exercises are hard — harder than on the homework, quizzes, or exams. That means that you should spend some time thinking and talking about them; they're designed to be solved in groups (the best way to learn mathematics). The problems are roughly in order of increasing difficulty. I don't expect any group to solve all of them.

Here's a hint: drawing pictures — e.g. sketching graphs of functions — will always make the problem easier.

Improper Integrals: infinite domains

Occasionally, and especially in physics problems, one wants to compute integrals over infinite domains. In these cases, the integral might be "infinity", but it might be finite. Usually, you can evaluate "improper" integrals (e.g. an integral over the domain $[1, \infty)$ just like any other integral: find an antiderivative, and then plug in the endpoints. Of course, "∞" is not a number, so plugging it in takes some skills. You learned these skills in 1A — "∞" really means a limit.

When you have to be careful is for domains that are infinite in both directions. In a situation like $\int_{-\infty}^{\infty}$, it's important to remember that "they are different ∞s".

1. To move an object against a force requires energy, also known as "work". If the object moves in one dimension from point a to point b against a force field $F(x)$, then the amount of work required is $W(a, b) = \int_a^b F(x)\,dx$.

   An object of mass m (e.g. a spaceship) is at distance R (e.g. the radius of the Earth) from a gravitating body (e.g. the Earth) of mass M. The force of gravity on the object when it is at distance x is $F(x) = GmM/x^2$, where G is a physical constant that is there only because humans don't work in units natural for doing gravitational physics (we work in units natural for everyday life instead).

   (a) Find the work required to move the object from its current positing R to a position infinitely far away from the planet.

(b) Remember that the kinetic energy of an object of mass m and velocity v is $mv^2/2$. For what v is the kinetic energy enough for the object to escape the gravitational pull? Recall that the "acceleration due to gravity" is $g = MG/R^2$; write your answer v as a function of g and R. The velocity v is called "the escape velocity at radius R".

Incidentally, the orbital velocity at radius R is $v = \sqrt{gR}$. How do the orbital and escape velocities compare?

2. The average speed of molecules in an ideal gas is

$$\bar{v} = \frac{4}{\sqrt{\pi}} \left(\frac{M}{2RT}\right)^{3/2} \int_0^\infty v^3 \, e^{-Mv^2/(2RT)} \, dv$$

where M is the molecular weight of the gas, T the temperature, and R is the ideal gas constant. Find $\bar{v}$. (Hint: first perform a u-substitution.)

3. Find all values of p for which $\int_1^\infty x^p \, dx$ converges. If $\int x^p \, dx$ converges, to what does it converge? (Your answer should, of course, be a function of p.)

4. Let n be a nonnegative integer. Using integration by parts, find $\int_0^\infty x^n \, e^{-x} \, dx$.

5. The "Laplace Transform" of a function $f(t)$, is the function of s given by

$$\mathcal{L}[f](s) = \int_0^\infty f(t) \, e^{-st} \, dt$$

if this integral converges. Find the Laplace transforms $\mathcal{L}[1](s)$, $\mathcal{L}[t](s)$, $\mathcal{L}[e^t](s)$, and $\mathcal{L}[t^n](s)$. (The last one requires your answer to the previous question.) What are the domains of these functions (for what s values do the integrals converge)?

Improper Integrals: infinite discontinuities

If a function has a finite discontinuity, integrating it is no problem: you break up the integral into pieces. But an infinite discontinuity can be deadly. Again, the answer to defining such integrals requires a limit. In general, if a function $f(x)$ on $[a, b]$ is continuous except for an infinite discontinuity at c, then we define

$$\int_a^b f(x) \, dx = \lim_{s \nearrow c} \int_a^s f(x) \, dx + \lim_{t \searrow c} \int_t^b f(x) \, dx.$$

The left-hand side is only defined if each limit on the right converges independently. Otherwise we say that the integral "diverges".

1. Find all values of p for which $\int_0^1 x^p\,dx$ converges. If $\int_0^1 x^p\,dx$ converges, to what does it converge? (Your answer should, of course, be a function of p.) Are there any values of p for which $\int_0^\infty x^p\,dx$ converges?

2. Is $\int_0^\pi \tan x\,dx$ well-defined (i.e. does the integral converge)? If so, to what?

3. Show that $\int_0^1 \ln x\,dx$ converges, and find the limit. More generally, use integration by parts to find $\int_0^1 (\ln x)^n\,dx$ for any nonnegative integer n.

A Comparison Test

If $0 \leq f(x) \leq g(x)$ on $[a, b]$ and $\int_a^b g(x)\,dx$ converges, then $\int_a^b f(x)\,dx$ also converges, and $0 \leq \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$. The functions are allowed to have (certain kinds of, but that's a technical issue that we will ignore in this course) discontinuities, and a and b are allowed to be $\pm\infty$.

1. Use the comparison test to decide if the following integrals converge:

$$\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx \qquad \int_1^\infty \frac{2+e^{-x}}{x}\,dx \qquad \int_0^{\pi/2} \frac{dx}{x\sin x} \qquad \int_1^\infty \frac{\arctan x}{x^2}\,dx$$

2. Evaluate the integral or show that it is divergent:

$$\int_0^1 \frac{t^2+1}{t^2-1}\,dt \qquad \int_2^6 \frac{y}{\sqrt{y-2}} \qquad \int_0^1 \frac{dx}{2-3x} \qquad \int_{-1}^1 \frac{x+1}{\sqrt[3]{x^4}}\,dx$$

3. In terms of b and c, evaluate

$$\int_{-\infty}^\infty \frac{dx}{x^2+bx+c}.$$

What conditions do you need to place on b and c to assure that this integral converges?

4. Find the value of the constant C for which the integral

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx$$

converges. Evaluate the integral for this value of C.