Math 1B Quiz 10

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/08Summer1B/

Tuesday, 5 August 2008

Name: ____ Score: /10

You have twenty minutes (plus the break) to complete the following closed-note open-chalkboard quiz. Partial credit will be awarded for correct work, and no points will be given for simply writing down the correct answer. Please box your final answers.

1. (5 pts) Express the following function as a power series:

$$\frac{1}{(1+x^2)^2}$$

$$\begin{aligned}
\frac{1}{1-x} &= \sum_{n=0}^{\infty} x^n \\
\frac{1}{1+x} &= \sum_{n=0}^{\infty} (-1)^n x^n \\
\frac{1}{(1+x)^2} &= -\frac{d}{dx}\left[\frac{1}{1+x}\right] \\
&= -\frac{d}{dx}\left[\sum_{n=0}^{\infty} (-1)^n x^n\right] \\
&= -\sum_{n=0}^{\infty} (-1)^n\, n x^{n-1} \\
&= -\sum_{n=0}^{\infty} (-1)^{n+1}\,(n+1) x^n \\
&= \sum_{n=0}^{\infty} (-1)^n\,(n+1)\,x^n \\
\frac{1}{(1+x^2)^2} &= \sum_{n=0}^{\infty} (-1)^n\,(n+1)\,(x^2)^n \\
&= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n}}
\end{aligned}$$

2. (0 pts) Favorite shade of dryer lint?

   Green.

3. (5 pts) Solve the differential equation by assuming that $y(x)$ can be expressed as a power series:

$$y'' - xy' - y = 0$$

$$\begin{aligned}
y &= \sum_{n=0}^{\infty} c_n x^n \\
xy' &= \sum_{n=0}^{\infty} n c_n x^n \\
y'' &= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n \\
0 &= y'' - xy - y \\
&= \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} c_n x^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - n c_n - c_n\right] x^n \\
&= \sum_{n=0}^{\infty} \left[(n+2)(n+1) c_{n+2} - (n+1) c_n\right] x^n \\
0 &= (n+2)(n+1) c_{n+2} - (n+1) c_n \\
c_{n+2} &= \frac{c_n}{n+2} \\
c_n &= \begin{cases} \frac{c_0}{2 \cdot 4 \cdot \ldots \cdot (2k)}, & n = 2k \text{ is even} \\ \frac{c_1}{1 \cdot 3 \cdot \ldots \cdot (2k+1)}, & n = 2k+1 \text{ is odd} \end{cases} \\
\text{Therefore } y &= \boxed{c_0 \sum_{k=0}^{\infty} \frac{x^{2k}}{2 \cdot 4 \cdot \ldots \cdot (2k)} + c_1 \sum_{k=0}^{\infty} \frac{x^{2k+1}}{1 \cdot 3 \cdot \ldots \cdot (2k+1)}}
\end{aligned}$$