Math 1B Quiz 7

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Name: ____ Score: /10

You have twenty minutes (plus the break) to complete the following closed-note open-chalkboard quiz. Partial credit will be awarded for correct work, and no points will be given for simply writing down the correct answer. Please box your final answers. Use the back of the page as necessary.

1. (0 pts) What kind of cookies should Theo bring to the midterm?

   Something homemade.

2. (10 pts) Solve the initial value problem:

$$4y'' - 4y' + y = 48t^2 e^{t/2} - 2t\cos\frac{t}{2}, \quad y(0) = 0, \quad y'(0) = 1$$

Hints: All coefficients in your final answer should be integers. This is two (or three, depending on how you count) problems rolled into one.

We begin by finding the general solution to the complementary equation:

$$\begin{aligned} 4y'' - 4y' + y &= 0 \\ 4r^2 - 4r + 1 &= 0 \\ (2r-1)^2 &= 0 \\ r &= 1/2 \\ y_c &= c_1 e^{t/2} + c_2 t e^{t/2} \end{aligned}$$

We now find a particular solution to $4y''-4y'+y = 48t^2e^{t/2}$. The linearly independent derivatives of $t^2e^{t/2}$ are $te^{t/2}$ and $e^{t/2}$; this has double overlap with $t^2e^{t/2}$. Thus we expect to need to multiply by t^2.

$$\begin{aligned} y_p &= At^4e^{t/2} + Bt^3e^{t/2} + Ct^2e^{t/2} \\ y_p' &= \frac{A}{2}t^4e^{t/2} + \left(4A + \frac{B}{2}\right)t^3e^{t/2} + \left(3B + \frac{C}{2}\right)t^2e^{t/2} + (2C)\,te^{t/2} \\ y_p'' &= \frac{A}{4}t^4e^{t/2} + \left(2A + \frac{4A}{2} + \frac{B}{4}\right)t^3e^{t/2} + \left(12A + \frac{3B}{2} + \frac{3B}{2} + \frac{C}{4}\right)t^2e^{t/2} + \end{aligned}$$

$$
\begin{aligned}
& \quad + (6B + C + C)\, te^{t/2} + (2C)\, e^{t/2} \\
48t^2 e^{t/2} &= y_p - 4y_p' + 4y_p'' \\
&= (A - 2A + A)\, t^4 e^{t/2} + (B - 2B + B - 16A + 16A)\, t^3 e^{t/2} + \\
& \quad + (C - 2C + C - 12B + 12B + 48A)\, t^2 e^{t/2} + (-8C + 8C + 24B)\, te^{t/2} + \\
& \quad + (8C)\, e^{t/2} \\
&= 0t^4 e^{t/2} + 0t^3 e^{t/2} + 48At^2 e^{t/2} + 24Bte^{t/2} + 8Ce^{t/2} \\
A &= 1 \\
B &= 0 \\
C &= 0 \\
y_p &= t^4 e^{t/2}
\end{aligned}
$$

With the same method, we find a particular solution to $4y'' - 4y' + y = -2t \cos t/2$*:*

$$
\begin{aligned}
y_p &= At \cos t/2 + Bt \sin t/2 + C \cos t/2 + D \sin t/2 \\
y_p' &= \frac{B}{2} t \cos t/2 - \frac{A}{2} \sin t/2 + \left(\frac{D}{2} + A \right) \cos t/2 + \left(-\frac{C}{2} + B \right) \sin t/2 \\
y_p'' &= -\frac{A}{4} t \cos t/2 - \frac{B}{4} \sin t/2 + \left(-\frac{C}{4} + 2\frac{B}{2} \right) \cos t/2 + \left(-\frac{D}{4} - 2\frac{A}{2} \right) \sin t/2 \\
-2t \cos t/2 &= y_p - 4y_p' + 4y_p'' \\
&= -2Bt \cos t/2 + 2A \sin t/2 + (-2D - 4A + 4B) \cos t/2 + (2C - 4B - 4A) \sin t/2 \\
-2 &= -2B \\
0 &= 2A \\
0 &= -2D - 4A + 4B \\
0 &= 2C - 4B - 4A \\
y_p &= t \sin t/2 + 2 \cos t/2 + 2 \sin t/2
\end{aligned}
$$

Thus, the general solution to the differential equation is

$$
y_g(t) = t^4 e^{t/2} + t \sin t/2 + 2 \cos t/2 + 2 \sin t/2 + c_1 e^{t/2} + c_2 te^{t/2}
$$

Plugging in $y(0) = 0$ *gives* $0 = 0 + 0 + 2 + 0 + c_1 + 0$. *Differentiating and plugging in* $y'(0) = 1$ *gives* $1 = 0 + 0 + 0 + 0 + 0 + 1 + c_1/2 + c_2 + 0$. *Thus* $c_1 = -2$ *and* $c_2 = 1$. *The final answer is*

$$
y(t) = t^4 e^{t/2} + t \sin t/2 + 2 \cos t/2 + 2 \sin t/2 - 2e^{t/2} + te^{t/2}
$$