Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. Introduce yourself to your new friends, and write all of your names at the top of the chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Introducing Derivatives

1. § If a ball is through into the air with a velocity of 40 ft/s, its heigh in feet t seconds later is given by $y = 40\text{ft/s}t - 16\text{ft/s}^2t^2$.
   (a) Find the average velocity for the time period beginning when $t = 2$ and lasting
      i. 0.5 seconds ii. 0.1 seconds iii. 0.05 seconds iv. 0.01 seconds
   (b) Estimate the instantaneous velocity when $t = 2$.
2. § The point $P = (1, \frac{1}{2})$ lies on the curve $\{y = x/(1 + x)\}$.
   (a) Let $Q = (x, x/(1 + x))$. In terms of x, find the slope of the secant line PQ. What is the domain of your function? Why?
   (b) Using your answer to part (a), find the value of the slop of PQ for the following values of x.
      i. 0.5 iii. 0.9 v. 0.99 vii. 0.999
      ii. 1.5 iv. 1.1 vi. 1.01 viii. 1.001

      You should report numerical answers rounded to a few decimal places. One way to do this is to use a calculator. Another way is find the reciprocal of the slope, and then use the fact that, if $|b| \ll |a|$ (b is much smaller than a), then
      $$\frac{1}{a+b} \approx \frac{1}{a} - \frac{b}{a^2} \tag{1}$$
   (c) If you plug $x = 1$ into your formula from part (a), what value of the slope would you get? Does this number agree with your answers to part (b)?
   (d) Find an equation of the tangent line to the curve at $P = (1, \frac{1}{2})$.
3. Let c be a real constant and $f(x)$ a function. How to the slopes of secant and tangent lines of $g(x) = f(x) + c$ compare to those of $f(x)$? What about $h(x) = f(x + c)$?
4. For each of the following functions, calculate the difference quotient $\dfrac{f(x+h) - f(x)}{h}$. Simplify your answers.
   (a) $f(x) = x$ (c) $f(x) = x^3$ (e) $f(x) = 1/x$ (g) $f(x) = 1/x^n$
   (b) $f(x) = x^2$ (d) $f(x) = x^n$ (f) $f(x) = 1/x^2$ (h) $f(x) = \sqrt{x}$

(We always let n be a fixed unknown positive integer.) What do difference quotients have to do with secant and tangent lines?

5. (a) Let $|h| \ll |x|$, so that h is much smaller than x. Simplify the difference quotient for the natural logarithm:

$$\frac{\ln(x+h) - \ln(x)}{h}$$

You should use rules about logarithms, and you should also use the approximation, true when c is very close to 0, that

$$\ln(1+c) \approx c. \tag{2}$$

(b) When $|h| \ll |x|$, simplify the difference quotient for the function e^x. Exponentiate equation (2) for a useful approximation.

$$e^c \approx 1 + c.$$

6. Let f and g be two functions, and a and b two numbers. Let l be the slope of the secant to f over the domain $[a, b]$ — i.e. l is the slope of the line that goes through the points $(a, f(a))$ and $(b, f(b))$ — and let m be the slope of the secant to g over the domain $[a, b]$. Let p be the slope of the secant to the function $h = f + g$ over $[a, b]$. Prove that $p = l + m$.

7. Prove equation (1):

   (a) In terms of a and b, evaluate the "error" of the approximation:

$$\left(\frac{1}{a+b}\right) - \left(\frac{1}{a} - \frac{b}{a^2}\right)$$

   (b) Argue that when b is much smaller than a, this error is very small.

8. In Math 1B we will develop tools to study approximations like (1). For example, we can generalize (1) to:

$$\frac{1}{a+b} \approx \frac{1}{a} - \frac{b}{a^2} + \frac{b^2}{a^3} - \cdots \pm \frac{b^n}{a^{n+1}} \tag{3}$$

where the last term is "+" if n is even, and "−" if n is odd. When $|b| < |a|$, for large enough n the fraction b^n/a^{n+1} is very small.

Let's prove equation (3), in the case when $|b| < |a|$:

   (a) Begin by multiplying both sides of the equation by $(a + b)$, and simplify the right-hand-side. What is the error of the new approximation?

   (b) You may use the following fact: if $|c| < 1$, and d is any positive number, then for some n, $|c^n| < d$. Show that for any d, there is some n such that the error from part (a) is less than d.

   (c) If n gets larger, what happens to the error?

Hard problems from previous days

9. § How is the graph of $y = f(|x|)$ related to the graph of $y = f(x)$? Sketch the graphs of $y = \sin|x|$, $y = \sqrt{|x|}$, and (most importantly for our class) $y = \ln|x|$.

10. Use algebra to show the shifting a graph by a units upward and then stretching vertically by a factor of b is the same as first stretching the graph vertically by a factor of b and then shifting upward by ab.