Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

More Limits

1. § Sketch the graph of an exmple of a function f such that $\lim_{x\to 3^+} f(x) = 4$, $\lim_{x\to 3^-} f(x) = 2$, $\lim_{x\to -2} f(x) = 2$, $f(3) = 3$, $f(-2) = 1$, and $\lim_{x\to 1} f(x) = +\infty$.

2. Last time, we defined two functions:

$$\lfloor x \rfloor = \text{greatest integer less than or equal to } x$$
$$\delta_{\mathbb{Z}}(x) = \begin{cases} 1, & \text{if } x \text{ is an integer} \\ 0, & \text{if } x \text{ is not an integer} \end{cases}$$

   We saw that $\lim_{x\to a} \lfloor x \rfloor$ exists only if a is not an integer, and $\lim_{x\to a} \delta_{\mathbb{Z}}(x) = 0$ for every a.

   (a) § Sketch the region in the plane defined by $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 1$.

   (b) Show that $\lfloor x \rfloor + \lfloor -x \rfloor = \delta_{\mathbb{Z}}(x) - 1$.

   (c) What does part (a) say about the Sum and Difference limit laws?

3. § Evaluate $\displaystyle\lim_{x\to 0} \frac{|2x-1| - |2x+1|}{x}$.

4. § Evaluate the limit, if it exists:

   (a) $\displaystyle\lim_{t\to -3} \frac{t^2-9}{2t^2+7t+3}$

   (b) $\displaystyle\lim_{h\to 0} \frac{(4+h)^2-16}{h}$

   (c) $\displaystyle\lim_{x\to -2} \frac{x+2}{x^3+8}$

   (d) $\displaystyle\lim_{t\to 9} \frac{9-t}{3-\sqrt{t}}$

   (e) $\displaystyle\lim_{x\to 7} \frac{\sqrt{x+2}-3}{x-7}$

   (f) $\displaystyle\lim_{x\to -4} \frac{4^{-1}+x^{-1}}{4+x}$

   (g) $\displaystyle\lim_{x\to 16} \frac{4-\sqrt{x}}{16x-x^2}$

   (h) $\displaystyle\lim_{t\to 0} \frac{1}{t\sqrt{t+1}} - \frac{1}{t}$

5. § In the theory of Special Relativity, the mass of a particle with velocity v is

$$m = \frac{m_0}{\sqrt{1-v^2/c^2}}$$

   where m_0 is the mass of the particle at rest and c is the speed of light.

   (a) What happens as $v \to c^-$?

   (b) Does it make sense to ask about $v \to c^+$?

6. § If $\lim_{x\to a} [f(x) + g(x)] = 2$ and $\lim_{x\to a} [f(x) - g(x)] = 1$, find $\lim_{x\to a} [f(x)\, g(x)]$. *Warning:* $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ might not exist.

7. § Give an example showing that $\lim_{x\to a}[f(x)\,g(x)]$ may exist even though neither $\lim_{x\to a} f(x)$ nor $\lim_{x\to a} g(x)$ exists.

8. True of False:
$$\text{If } \lim_{x\to 1}[f(x)]^2 = 4, \text{ then } \lim_{x\to 1} f(x) = 2$$
If true, prove it. If false, find a counterexample.

9. § If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim_{x\to 4} f(x)$.

10. § Prove that $\lim_{x\to 0^+} \sqrt{x}e^{\sim(\pi/x)} = 0$.

11. Sketch a graph of $y = \dfrac{x^2 - 4}{|x+2|}$.

12. § Find the limit, or prove it does not exist.

    (a) $\lim\limits_{x\to 3}(2x + |x-3|)$ (b) $\lim\limits_{x\to -6}\dfrac{2x+12}{|x+6|}$ (c) $\lim\limits_{x\to 0.5^-}\dfrac{2x-1}{|2x^3 - x^2|}$ (d) $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{|x|}\right)$

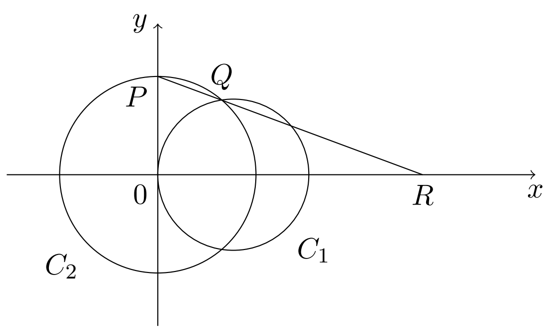
13. § Use the limit laws to prove: if p is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

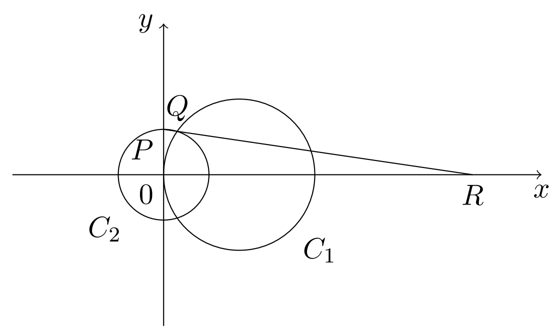
14. § Use the limit laws and the previous exercise to prove: if r is a rational function and a is in the domain of r, then $\lim_{x\to a} r(x) = r(a)$. Give an example to explain why the condition "a is in the domain of r" is necessary.

15. § (a) Evaluate $\lim\limits_{x\to 2}\dfrac{\sqrt{6-x}-2}{\sqrt{3-x}-1}$. (b) Evaluate $\lim\limits_{x\to 1}\dfrac{\sqrt[3]{x}-1}{\sqrt{x}-1}$.

16. § Is there a number a so that $\lim\limits_{x\to -2}\dfrac{3x^2 + ax + a + 3}{x^2 + x - 2}$ exists? If so, find a and the limit.

17. § Let C_1 be a "fixed" circle $C_1 = \{(x-1)^2 + x^2 = 1\}$ centered at $(1, 0)$ with radius 1, and let C_2 be a "shrinking" circle $C_2 = \{x^2 + y^2 = r^2\}$ of radius r centered that the origin. (See diagram below.) Let $P = (0, r)$ be the point where C_2 intersects the positive y-axis, and Q the upper point of intersection of the two circles, and let R be the point of intersection of the line PQ with the x-axis. What happens to R as C_2 shrinks, i.e. what is $\lim_{r\to 0} R$?





Hard problems from previous days

18. Use algebra to show the shifting a graph by a units upward and then stretching vertically by a factor of b is the same as first stretching the graph vertically by a factor of b and then shifting upward by ab.