Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Limits and Continuity

1. *[This exercises is based on suggestions from Prof. Zvezdelina Stankova.]*
   (a) State the precise definition of a limit.
   (b) Use the definition of a limit to prove that $\lim_{x\to 1} 2x+3$ is not equal to 6.
   (c) § Use the definition of a limit to prove that $\lim_{x\to 1} 2x+3$ equals 5.

2. *[This exercises is based on suggestions from Prof. Zvezdelina Stankova.]* The definition of a limit sets up a game: on your opening move, you claim that $\lim_{x\to a} f(x) = L$. Then I suggest an ϵ, and you respond with a δ that beats my ϵ. I give a new ϵ, you a new δ, etc. To prove your claim is the same as to present a winning strategy for any ϵ I may make.
   (a) What is the requirement on δ in order for it to win against ϵ?
   (b) Are you allowed to respond with a δ that is smaller than strictly necessary, or must you always respond with the largest possible δ?
   (c) Let's say I play a particular ϵ, and you win with a δ. If I play a larger ϵ, does your same δ necessarily still win?

3. State the precise definition of an infinite limit. Prove that $\displaystyle\lim_{x\to 1} \frac{-2}{(x-1)^4} = -\infty$.

4. § Prove the statement using the ϵ, δ definition of limit:
   (a) $\lim\limits_{x\to 6} \left(\frac{x}{4}+3\right) = \frac{9}{2}$
   (b) $\lim\limits_{x\to -1.5} \frac{9-4x^2}{3+2x} = 6$
   (c) $\lim\limits_{x\to 0} |0| = 0$
   (d) $\lim\limits_{x\to 9^-} \sqrt[4]{9-x} = 0$
   (e) $\lim\limits_{x\to 3} \left(x^2+x-4\right) = 8$
   (f) $\lim\limits_{x\to 2} x^3 = 8$

5. § For what value of the constant c is the given function f continuous on $(-\infty,\infty)$?
$$f(x) = \begin{cases} cx^2+2x & \text{if } x<2 \\ x^3-cx & \text{if } x\geq 2 \end{cases}$$

6. Draw pictures that illustrate the following terms: (a) removable discontinuity, (b) jump discontinuity, (c) infinite discontinuity, (d) essential discontinuity.

7. § Let f be the function defined by
$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ 1 & \text{if } x \text{ is rational} \end{cases}$$

(a) Show that $\lim_{x\to 0} f(x)$ does not exist.

(b) At what values of x if $f(x)$ continuous?

(c) Let $g(x) = x\,f(x)$. At what values of x is $g(x)$ continuous?

More Limits from last time

8. § If $4x - 9 \le f(x) \le x^2 - 4x + 7$ for $x \ge 0$, find $\lim_{x\to 4} f(x)$.

9. § Prove that $\lim_{x\to 0^+} \sqrt{x}\, e^{\sin(\pi/x)} = 0$.

10. Sketch a graph of $y = \dfrac{x^2 - 4}{|x+2|}$.

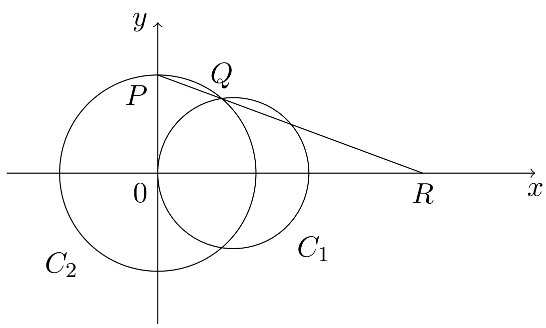
11. § Find the limit, or prove it does not exist.

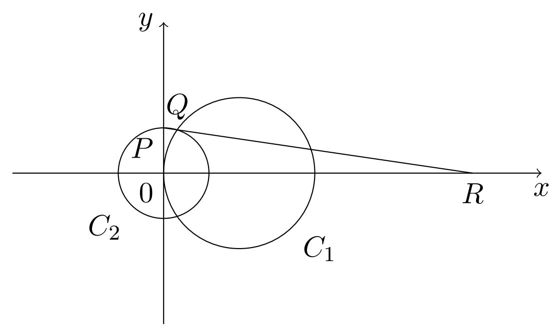
    (a) $\displaystyle\lim_{x\to 3} (2x + |x-3|)$ (b) $\displaystyle\lim_{x\to -6} \frac{2x+12}{|x+6|}$ (c) $\displaystyle\lim_{x\to 0.5^-} \frac{2x-1}{|2x^3 - x^2|}$ (d) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{|x|}\right)$

12. § (a) Evaluate $\displaystyle\lim_{x\to 2} \frac{\sqrt{6-x} - 2}{\sqrt{3-x} - 1}$. (b) Evaluate $\displaystyle\lim_{x\to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1}$.

13. § Is there a number a so that $\displaystyle\lim_{x\to -2} \frac{3x^2 + ax + a + 3}{x^2 + x - 2}$ exists? If so, find a and the limit.

14. § Let C_1 be a "fixed" circle $C_1 = \{(x-1)^2 + x^2 = 1\}$ centered at $(1,0)$ with radius 1, and let C_2 be a "shrinking" circle $C_2 = \{x^2 + y^2 = r^2\}$ of radius r centered that the origin. (See diagram below.) Let $P = (0, r)$ be the point where C_2 intersects the positive y-axis, and Q the upper point of intersection of the two circles, and let R be the point of intersection of the line PQ with the x-axis. What happens to R as C_2 shrinks, i.e. what is $\lim_{r\to 0} R$?





Abstract proofs

15. Use algebra to show the shifting a graph by a units upward and then stretching vertically by a factor of b is the same as first stretching the graph vertically by a factor of b and then shifting upward by ab.

16. § Use the limit laws to prove: if p is a polynomial, then $\lim_{x\to a} p(x) = p(a)$.

17. § Use the limit laws and the previous exercise to prove: if r is a rational function and a is in the domain of r, then $\lim_{x\to a} r(x) = r(a)$. Give an example to explain why the condition "a is in the domain of r" is necessary.