Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard.

Some logic

1. Prove that for every real number A, there exists a real number B such that $B > A$.

2. Let $\diamond$ be a relationship between numbers: for any given real numbers a and b, either $a \diamond b$ is true, or it's not. For example, $\diamond$ might be $\leq$, or $=$, or $>$, or $\neq$, or it might be the relationship that "$a \diamond b$" means "$a = 3$".

   (a) List a few more examples of relationships. Is there a relationship that is always true? Never true?

   (b) If $\diamond$ is a relationship, let "$a\blacksquare b$" mean "it is not true that $a \diamond b$". Is $\blacksquare$ a relationship?

   (c) Give two examples of relationships $\diamond$ so that $a \diamond b$ if and only if $b \diamond a$.

   (d) Is there a relationship so that EITHER $a \diamond b$ OR $b \diamond a$, BUT NOT BOTH? If so, give an example. What about a relationship so that: if $a \neq b$, then EITHER $a \diamond b$ OR $b \diamond a$?

   (e) Give an example of a relationship $\diamond$ so that:

   > For every real number a, there exists a real number b such that $a \diamond b$.

   Prove that your relationship has the desired property.

   (f) Give an example of a relationship $\diamond$ so that the following is FASLE:

   > For every real number a, there exists a real number b such that $a \diamond b$.

   Prove your example works.

3. Let $\diamond$ be a relationship. Assume that for every real number a, there exists a real number b (which may depend on a) such that $a \diamond b$. Which of the following are NECESSARILY TRUE? Which are NECESSARILY FALSE? If a statement is neither necessarily true nor necessarily false, give an example for which the statement is false, and one for which it is true.

   (A statement is "necessarily true" if it follows from the assumptions: in our case, this means that any $\diamond$ satisfying the assumptions will also satisfy any necessarily true statement. There may be relationships satisfying the assumptions and also satisfying statements that are not "necessarily true". A statement is "necessarily false" if from the assumptions you can prove it is false: no $\diamond$ satisfying the assumptions also satisfies the statement.)

   (a) If a is a real number, then $a \diamond b$ for every real number b.

   (b) If b is a real number, then $a \diamond b$ for every real number a.

   (c) If a is a real number, then there exists a real number b so that $a \diamond b$.

   (d) If b is a real number, then there exists a real number a so that $a \diamond b$.

   (e) There exists a real number a so that for every real number b, $a \diamond b$.

   (f) There exists a real number b so that for every real number a, $a \diamond b$.

(g) There exists a real number a so that there exists a real number b with $a \diamond b$.

(h) There exists a real number b so that there exists a real number a with $a \diamond b$.

(i) There exists a real number a so that for every real number b, it is not true that $a \diamond b$.

(j) There exists a real number b so that for every real number a, it is not true that $a \diamond b$.

(k) For every real number a, there exists a real number b so that it is not true that $a \diamond b$.

(l) For every real number b, there exists a real number a so that it is not true that $a \diamond b$.

(m) There exist real numbers a and b so that it is not true that $a \diamond b$.

(n) For all real numbers a and b, it is not true that $a \diamond b$.

4. Let $\diamond$ be a relationship. How would you prove that "For every real number a, there exists a real number b such that $a \diamond b$" is FALSE?

5. Let $\diamond$ be a relationship such that "For every real number a, there exists a real number b such that $a \diamond b$" is FALSE. Which of the statements in exercise 3 are NECESSARILY TRUE? Which are NECESSARILY FALSE?

6. Let $a \diamond b$ be the following relationship: "If $x < a$, then $x^2 < b$." Is it true that for any real number a, there exists a real number b such that $a \diamond b$? Prove your answers.

7. Let $a \diamond b$ be the following relationship: "If $x^2 < a$, then $x < b$." Is it true that for any real number a, there exists a real number b such that $a \diamond b$? Prove your answer.

8. Let $\heartsuit$ and $\clubsuit$ be two relationships. How would you prove that "$a \heartsuit b$ implies $a \clubsuit b$"? How would you disprove it?

9. Let $\heartsuit$ and $\clubsuit$ be two relationships. How would you prove or disprove: "If $x \heartsuit a$ then $x \clubsuit b$."

10. Let $\heartsuit$ and $\clubsuit$ be two relationships. Define $\diamond$ by "$a \diamond b$" means "If $x \heartsuit a$, then $x \clubsuit b$".

    (a) For a given pair a and b, how would you prove $a \diamond b$?

    (b) How would you prove that: For any real number a, there exists a real number b such that $a \diamond b$?

    (c) How would you prove that: For any real number b, there exists a real number a such that $a \diamond b$?

    (d) How would you prove that: There exists a real number b such that for any real number a, it is not true that $a \diamond b$?

    (e) How would you prove that: It is not true that for any real number b, there exists a real number a such that $a \diamond b$?

11. Let f be a function, and K and L two numbers. Let $x \heartsuit a$ mean that $0 < |x - K| < a$. Let $x \clubsuit b$ mean that $|f(x) - L| < b$. Let $a \diamond b$ mean that "$x \heartsuit a$ implies $x \clubsuit b$". What is another way of writing the following statement:

    (∗) For any positive real number b, there exists a positive real number a such that $a \diamond b$.

12. In problem 11, let $f(x) = x^2 - 1$, and $K = 2$. If $L = 5$, is Statement (∗) true or false? If $L = 3$, is Statement (∗) true or false? Prove your answers.