Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Warm-up from previous days

1. § Evaluate $\lim_{x\to 2} \arctan\left(\frac{x^2-4}{3x^2-6x}\right)$.

2. Let $\heartsuit$ and $\clubsuit$ be two relationships. Define $\diamond$ by: "$a \diamond b$" means "If $x \heartsuit a$, then $x \clubsuit b$".
   (a) What would be required to prove: "There exists an x such that $x \heartsuit 1$"? What would be required to disprove it?
   (b) What would be required to prove: "For every x, $x \heartsuit 1$"? What would be required to disprove it?
   (c) What would be required to prove: "For every a there exists an x such that $x \heartsuit a$"? What would be required to disprove it?
   (d) Given numbers a and b, what would be required to prove: "$a \diamond b$"?
   (e) What would be required to prove: "For every b there exists an a such that $a \diamond b$"?
   (f) What would be required to prove: "It is not true that for every b there exists an a such that $a \diamond b$"?

3. § Prove the statement using the ϵ, δ definition of limit:
   (a) $\lim_{x\to 6}\left(\frac{x}{4}+3\right) = \frac{9}{2}$ (b) $\lim_{x\to 0} |x| = 0$ (c) $\lim_{x\to 2} x^3 = 8$

4. § For what value of the constant c is the given function f continuous on $(-\infty, \infty)$?

$$f(x) = \begin{cases} cx^2 + 2x & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Intermediate Value Theorem and infinite limits

5. § Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval:
   (a) $x^4 + x - 3 = 0$, $x \in (1, 2)$ (b) $\cos x = x$, $x \in (0, 1)$ (c) $\ln x = e^{-x}$, $x \in (1, 2)$

6. Use the infinite limit laws and the Intermediate Value Theorem to prove the following result: If a polynomial $p(x)$ has odd degree (so the highest exponent appearing with a non-zero coefficient in $p(x)$ is odd), then $p(x)$ has at least one real root.

7. (a) § Can the graph of a function intersect a vertical asymptote? Can it intersect a horizontal asymptote? Justify your answers, either with a proof or a counterexample.
   (b) § How many horizontal asymptotes can the graph of a function have? Illustrate all the possibilities.

8. Sketch the graph of an example of a function f such that: $\lim_{x\to 0^+} f(x) = -\infty$, $\lim_{x\to\infty} f(x) = 2$, $f(3) = 3$, and f is odd.

9. § Evaluate the limit and justify each step by indicating the appropriate properties of limit:
$$\lim_{x\to\infty} \frac{3x^2 - x + 4}{2x^2 + 5x - 8}$$

10. The following equation is usually nonsensical:
$$\frac{\infty^2}{\infty} = \infty$$
   (a) If we interpret the symbol "∞" as meaning "the limit of any function that goes to infinity", give an example showing that the above equation is not necessarily true.
   (b) If we interpret the symbol "∞" as meaning something like "the limit of x as $x \to \infty$", explain whether the above equation is justified.

11. In this course, there are two "infinities": $+\infty$ and $-\infty$. Let's replace those with a different notion: let's introduce the symbol Ω to mean $\lim_{x\to 0} 1/x$. What properties does Ω have? Is it positive? Negative? What should we make of the expressions $\Omega + 2$, $\Omega \times 3$, $-1/\Omega$, $\Omega + \Omega$, and Ω^2?

Hard problems from previous days

12. Use algebra to show the shifting a graph by a units upward and then stretching vertically by a factor of b is the same as first stretching the graph vertically by a factor of b and then shifting upward by ab.

13. § Let C_1 be a "fixed" circle $C_1 = \{(x-1)^2 + x^2 = 1\}$ centered at $(1, 0)$ with radius 1, and let C_2 be a "shrinking" circle $C_2 = \{x^2 + y^2 = r^2\}$ of radius r centered that the origin. (See diagram below.) Let $P = (0, r)$ be the point where C_2 intersects the positive y-axis, and Q the upper point of intersection of the two circles, and let R be the point of intersection of the line PQ with the x-axis. What happens to R as C_2 shrinks, i.e. what is $\lim_{r\to 0} R$?