Math 1A: Discussion Exercises

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/09Spring1A/

Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Warm-up from previous days

1. Be sure to understand the statements and proofs of the two most important applications of the Intermediate Value Theorem:
   (a) Any continuous function from the interval to itself has a fixed point.
   (b) Any polynomial of odd degree has at least one real root.

To ∞ and beyond

2. § Find the limit (possibly $\pm\infty$), or explain why it does not exist:
   (a) $\displaystyle\lim_{t\to-\infty} \frac{t^2+2}{t^3+t^2-1}$
   (b) $\displaystyle\lim_{x\to\infty} \frac{x+2}{\sqrt{9x^2+1}}$
   (c) $\displaystyle\lim_{x\to\infty} \cos x$
   (d) $\displaystyle\lim_{x\to\infty} \sqrt{x^2+1}$
   (e) $\displaystyle\lim_{x\to\infty} \frac{x^3-2x+3}{5-2x^2}$
   (f) $\displaystyle\lim_{x\to\infty} \arctan\left(x^2-x^4\right)$

3. § Find the right- and left-hand limits of:
$$\lim_{x\to\frac{\pi}{2}} e^{\tan x}$$
   Explain why they don't agree, and graph the function $y = e^{\tan x}$.

4. § Find the limit
$$\lim_{x\to\infty}\left(\sqrt{3x^2+8x+6}-\sqrt{3x^2+3x+1}\right)$$
   by multiplying and dividing by the conjugate.

5. § Evaluate $\displaystyle\lim_{x\to\infty}\frac{\sin x}{x}$ (hint: Squeeze Theorem). How many times does the function cross the horizontal asymptote (hint: solve an equation)?

6. The following equation is usually nonsensical:
$$\frac{\infty^2}{\infty} = \infty$$
   (a) If we interpret the symbol "∞" as meaning "the limit of any function that goes to infinity", give an example showing that the above equation is not necessarily true.

(b) If we interpret the symbol "∞" as meaning something like "the limit of x as $x \to \infty$", explain whether the above equation is justified.

7. In this course, there are two "infinities": $+\infty$ and $-\infty$. Let's replace those with a different notion: let's introduce the symbol Ω to mean $\lim_{x\to 0} 1/x$. What properties does Ω have? Is it positive? Negative? What should we make of the expressions $\Omega + 2$, $\Omega \times 3$, $-1/\Omega$, $\Omega + \Omega$, and Ω^2?

Introducing derivatives

8. Write the definition of the slope $f'(a)$ of the tangent line to the curve $y = f(x)$ at $x = a$.

9. § Use the limit definition to find the slope of the tangent line to the curve at the given point:

   (a) $y = \dfrac{x-1}{x-2}$, $(3, 2)$ (b) $y = \sqrt{x}$, $(1, 1)$ (c) $y = x^2$, (a, a^2) (d) $y = x^n$, (a, a^n)

10. Let $h(x) = f(x) + g(x)$. Use the definition of derivative to prove that $h'(a) = f'(a) + g'(a)$.

Hard problems from previous days

11. Use algebra to prove that shifting a graph by a units upward and then stretching vertically by a factor of b is the same as first stretching the graph vertically by a factor of b and then shifting upward by ab.

12. § Let C_1 be a "fixed" circle $C_1 = \{(x-1)^2 + x^2 = 1\}$ centered at $(1, 0)$ with radius 1, and let C_2 be a "shrinking" circle $C_2 = \{x^2 + y^2 = r^2\}$ of radius r centered that the origin. (See diagram below.) Let $P = (0, r)$ be the point where C_2 intersects the positive y-axis, and Q the upper point of intersection of the two circles, and let R be the point of intersection of the line PQ with the x-axis. What happens to R as C_2 shrinks, i.e. what is $\lim_{r\to 0} R$?