Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Derivatives

1. § For each of the following functions, find $f'(a)$. You may use only: the definition of the derivative, and Derivative Laws allowed on next week's midterm (sum, difference, constant multiple, power, exponential).

   (a) $f(t) = \dfrac{2t+1}{t+3}$ (b) $f(x) = \dfrac{1}{\sqrt{x+2}}$ (c) $\sqrt{3x+1}$ (d) $f(x) = 3-2x+4x^2$

2. § Make a careful sketch of the graph of $y = \sin x$, and sketch the graph of the derivative $\sin' x$. In particular, what are the zeros of $\sin'$, and where is it positive and negative. Can you guess the formula for $\sin'$ based on the graph?

3. What is the domain of the function $f(x) = \sqrt{x}$? What is the domain of its derivative $f'(x)$?

4. § Suppose that f is a function with the property that $|f(x)| \leq x^2$ for all x. Show that $f(0) = 0$. Then show that $f'(0) = 0$.

5. § Differentiate the following functions, using only the Derivative Laws allowed on the midterm.

   (a) $f(x) = \sqrt{30}$ (d) $f(x) = 5e^x + 3$ (g) $F(x) = \left(\frac{1}{2}x\right)^5$

   (b) $f(t) = \frac{1}{2}t^6 - 3t^4 + t$ (e) $y = \sqrt[3]{x}$ (h) $y = ax^2 + bx + c$

   (c) $g(t) = \dfrac{1}{4}\left(t^4 + 8\right)$ (f) $h(x) = \dfrac{x^2 - 2\sqrt{x}}{x}$ (i) $v = \left(\sqrt[3]{x} + \dfrac{1}{\sqrt{x}}\right)^2$

6. § Find equations for the tangent line and the normal line to the curve $y = (1+2x)^2$ at the point $(1, 9)$.

7. § Graph the function $f(x) = x + \dfrac{1}{x}$, and also find and graph the derivative $F'(x)$. Are your graphs consistent?

8. § Find all derivatives of the function $f(x) = x^4 - 3x^3 + 16x$. I.e. find the first derivative, the second derivative, etc., until you get some derivative that is identically 0.

9. What happens if you take the function $1/x$ and start differentiating? Does the sequence of functions ever stop, in the sense of eventually becoming identically zero? Justify your answer.

10. § For what values of x does the graph of $y = x^3 + 3x^2 + x + 3$ have a horizontal tangent?

11. § Show that the curve $y = 6x^3 + 5x - 3$ has no tangent line with slope 4.

12. § Find an equation of the tangent line to the curve $y = x\sqrt{x}$ that is parallel to the line $y = 1 + 3x$.

13. § Find equations of both lines that are tangent to the curve $y = 1 + x^3$ and are perpendicular to the line $x + 12y = 1$.

14. § Draw a picture to show that there are two tangent lines to the parabola $y = x^2$ that pass through the point $(0, -4)$. Find the coordinates of the points where these tangent lines meet the parabola.

15. (a) § Find equations of both lines through the point $(2, -3)$ that are tangent to the parabola $y = x^2 + x$.
    (b) § Provide two proofs, one using algebra and the other using a graph, to show that there is no line through the point $(2, 7)$ that is tangent to the parabola.

16. § Find a second-degree (i.e. quadratic) polynomial P such that $P(2) = 5$, $P'(2) = 3$, and $P''(2) = 2$.

17. § Let
$$f(x) = \begin{cases} x^2 & \text{if } x \le 2 \\ mx + b & \text{if } x > 2 \end{cases}$$
For what values of m and b is $f(x)$ differentiable everywhere?

18. Find the derivative of the function $f(x) = x \cdot |x|$. Be sure to specify at what points f is differentiable.

Product and Quotient Rules

19. § Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find $(fg)'(5)$, $(f/g)'(5)$, and $(g/f)'(5)$.

20. § Differentiate. You may use the product and quotient rules.
    (a) $(x^3 + 2x)e^x$
    (b) $(u^{-2} + u^{-3})(u^5 - 2u^2)$
    (c) $\dfrac{x+1}{x^3 + x - 2}$
    (d) $\dfrac{t}{(t-1)^2}$
    (e) $\dfrac{2t}{4+t^2}$
    (f) $\dfrac{ax+b}{cx+d}$

21. § Find $f'(x)$ and $f''(x)$:
    (a) $f(x) = x^4 e^x$
    (b) $f(x) = x^{5/2} e^x$
    (c) $f(x) = \dfrac{x^2}{1+2x}$

22. § How many tangent lines to the curve $y = x/(x+1)$ pass through the point $(1, 2)$? At what points do these tangent lines touch the curve?

23. § Use the Product Rule twice to prove that if f, g, h are differentiable, then $(fgh)' = f'gh + fg'h + fgh'$. Then take $f = g = h$ to show that $\frac{d}{dx}\big[f(x)\big]^3 = 3\big[f(x)\big]^2 f'(x)$, and use this to differentiate $y = e^{3x}$.

24. § If f and g are differentiable, show that $(fg)'' = f''g + 2f'g' + fg''$. Find similar formulas for $(fg)'''$ and $(fg)^{(4)}$. Do you notice a pattern? Guess a formula for $(fg)^{(n)}$.