Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Trig functions

1. § Differentiate.

   (a) $2\csc x + 5\cos x$
   (b) $4\sec t + \tan t$
   (c) $e^u(\cos u + cu)$
   (d) $(1 - \sec x)/\tan x$
   (e) $\csc\theta(\theta + \cot\theta)$
   (f) $xe^x \csc x$

2. § Find an equation of the tangent line to the curve at the given point::

   (a) $y = \sec x$, $(\pi/3, 2)$
   (b) $y = e^x \cos x$, $(0, 1)$
   (c) $y = x + \cos x$, $(0, 1)$

3. § If $f(x) = e^x \cos x$, find $f'(x)$ and $f''(x)$.

4. Either by remembering the proof from the textbook, or by recognizing them as derivatives, find the following limits:

$$\lim_{x\to 0} \frac{\sin x}{x} \qquad \lim_{x\to 0} \frac{\cos x - 1}{x}$$

5. § Using the previous problem, find the following limits:

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{x}$
   (b) $\displaystyle\lim_{t\to 0} \frac{\tan 6t}{\sin 2t}$
   (c) $\displaystyle\lim_{\theta\to 0} \frac{\cos\theta - 1}{\sin\theta}$
   (d) $\displaystyle\lim_{x\to 0} \frac{\sin(\cos x)}{\sec x}$
   (e) $\displaystyle\lim_{x\to 0} \frac{\sin(x^2)}{x}$
   (f) $\displaystyle\lim_{x\to 0} \frac{\sin x}{x + \tan x}$

6. § Find the limits:

$$\lim_{x\to \pi/4} \frac{1 - \tan x}{\sin x - \cos x} \qquad \lim_{x\to 1} \frac{\sin(x-1)}{x^2 + x - 2}$$

7. (a) § A mass on a spring oscillates horizontally on a level, frictionless surface. Its equation of motion is $x(t) = 2\sin 3t$. Find the velocity $v(t)$ and acceleration $a(t)$ of the mass as functions of time.

   (b) What is the relationship between $x(t)$ and $a(t)$?

   (c) Show that for any numbers A and B, the equation of motion $x(t) = A\sin 3t + B\cos 3t$ satisfies the same relationship as you found in part (b).

8. (a) § Prove, using the definition of derivative and the sum-of-angles formulae, that $\cos' = \sin$.

(b) § Prove, using the formulas for the derivatives of sin and cos and also the quotient rule, the formulae for the derivatives of csc, sec, and cot.

9. What's the derivative of $\sin^2 x$? What's the derivative of $\cos^2 x$? What happens when you add them together and why?

10. Find numbers A and B so that

$$\frac{d}{dx}\left[Ae^x \cos x + Be^x \sin x\right] = e^x \cos x$$

Harder questions on earlier material

11. Write out the first few derivatives $(f, f', f'', \ldots)$ of $f(x) = xe^x$. Do you notice a pattern?

12. (a) Prove that if p is a polynomial of degree n, then the derivative of $p(x)\,e^x$ is $q(x)\,e^x$, where q is also a polynomial of degree n.
    (b) Let $f(x) = p(x)\,e^x$ where p is a polynomial. What is $\lim\limits_{x\to-\infty} f^{(n)}(x)$?

13. Let r be a rational function, so that $r(x) = p(x)/q(x)$ for some polynomials p and q. Define the *degree* of r to be $\deg p - \deg q$, where $\deg p$ is the degree of p, i.e. the highest power of a non-zero term in p. Prove that r', the derivative of r, is a rational function, and prove that $\deg r' = \deg r - 1$.

14. (a) Use the definition of derivative to prove the product rule.
    (b) Use the product rule to prove the quotient rule.
    (c) Let p be a polynomial. Use the product rule, but not the chain rule, to prove that $\frac{d}{dx}\left[p(q(x))\right] = p'(q(x))\,q'(x)$.