Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Chain Rule

1. § Differentiate.
   (a) $(4x - x^2)^{100}$
   (b) $(1 + x^4)^{3/2}$
   (c) $\sqrt[3]{1 + \tan t}$
   (d) $\dfrac{(y-1)^4}{(y^2+2y)^5}$
   (e) $a^3 + \cos^3 x$
   (f) $(t^4 - 1)^3(t^3 + 1)^4$
   (g) 10^{1-x^2}
   (h) $\dfrac{e^u - e^{-u}}{e^u + e^{-u}}$
   (i) $\tan^2(3\theta)$
   (j) $e^{k \tan \sqrt{x}}$
   (k) $\sin(\sin(\sin x))$
   (l) $\left(\dfrac{y^2}{y+1}\right)^2$

2. § Suppose that f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for $F'(x)$ and $G'(x)$.

3. § For what values of r does the function $y = e^{rx}$ satisfy the differential equation $y'' + 5y' - 6y = 0$?

4. § Find the 50th derivative of $y = \cos 2x$. Find the 1000th derivative of $f(x) = xe^{-x}$.

5. § Air is being pumped into a spherical balloon. At any time t, the volume of the balloon is $V(t)$ and the radius is $r(t)$. What do the derivatives dV/dr and dV/dt represent? What is the relationship between dV/dt, r, and dr/dt?

6. § Use the chain rule to prove that the derivative of an even function is an odd function, and that the derivative of an odd function is an even function.

7. § If n is a positive integer, prove that:

$$\frac{d}{dx}\big(\sin^n x \, \cos nx\big) = n \sin^{n-1} x \, \cos\big((n+1)x\big)$$

   Find a similar formula for $\frac{d}{dx}\big(\cos^n x \, \cos nx\big)$.

8. The Leibniz notation makes the chain rule very natural looking:

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

However, the corresponding formula for the second derivative — $\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\frac{du^2}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2$ — is false. Instead, prove the following "chain rule for second derivatives", by using the chain and product rules:

$$\frac{d^2y}{dx^2} = \frac{d^2y}{du^2}\left(\frac{du}{dx}\right)^2 + \frac{dy}{du}\frac{d^2u}{dx^2}$$

9. (a) Let f be a differentiable function, such that $f'(x) > 0$ for all x. By drawing a graph, show that f is an increasing and hence one-to-one.
   (b) Let g be the inverse of f, so that $f(g(x)) = g(f(x)) = x$. Use the chain rule to find the derivative of g in terms of the derivative of f.
   (c) Show that $\ln' x$ — the derivative of $\ln x$ — is $1/x$.

Harder questions on earlier material

10. Write out the first few derivatives $(f, f', f'', \dots)$ of $f(x) = xe^x$. Do you notice a pattern?

11. (a) Prove that if p is a polynomial of degree n, then the derivative of $p(x)\,e^x$ is $q(x)\,e^x$, where q is also a polynomial of degree n.
    (b) Let $f(x) = p(x)\,e^x$ where p is a polynomial. What is $\lim_{x\to-\infty} f^{(n)}(x)$?

12. Let r be a rational function, so that $r(x) = p(x)/q(x)$ for some polynomials p and q. Define the *degree* of r to be $\deg p - \deg q$, where $\deg p$ is the degree of p, i.e. the highest power of a non-zero term in p. Prove that r', the derivative of r, is a rational function, and prove that $\deg r' = \deg r - 1$.

13. (a) Use the definition of derivative to prove the product rule.
    (b) Use the product rule to prove the quotient rule.
    (c) Let p be a polynomial. Use the product rule, but not the chain rule, to prove that $\frac{d}{dx}\left[p(q(x))\right] = p'(q(x))\,q'(x)$.

    In fact, from just the product rule, you can prove the chain rule provided that the outer function is a rational function (ratio of two polynomials).

14. What's the derivative of $\sin^2 x$? What's the derivative of $\cos^2 x$? What happens when you add them together and why?

15. Find numbers A and B so that

$$\frac{d}{dx}\left[Ae^x \cos x + Be^x \sin x\right] = e^x \cos x$$

16. Let $f(x) = a^x$. Then $f'(0) = \ln a$. (Why?) Let's say we didn't know that. Define the function $\ell(a)$ for $a > 0$ by $\ell(a) = \frac{d}{dx}a^x\Big|_{x=0}$. Use the product rule to show directly that $\ell(ab) = \ell(a) + \ell(b)$. Use the quotient rule to show directly that $\ell(a/b) = \ell(a) - \ell(b)$. Show directly that $\ell(1) = 0$.