Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Logarithms

1. § Differentiate.
   (a) $\ln \sqrt[5]{x}$
   (b) $\dfrac{1+\ln t}{1-\ln t}$
   (c) $1/\ln(x)$
   (d) $\ln \sqrt{\dfrac{a^2-x^2}{a^2+x^2}}$
   (e) $\left[\ln(1+e^x)\right]^2$
   (f) $\log_2(e^{-x}\cos \pi x)$

2. Compute the derivative of $\ln(x^n)$ in two different ways: by using logarithm identities, and by using the chain rule.

3. § Find y' if $x^y = y^x$.

4. Prove the product rule in the following manner: (a) Let f and g be two functions, and consider the function $\ln\big(f(x)\,g(x)\big)$. Use logarithm rules to rewrite it as the sum of two functions. (b) Take derivatives of each side of your logarithm identity, using the chain rule. (c) Multiply both sides by the common denominator.

5. § Compute y' by first simplifying $\ln y$ and then using the fact that $(\ln y)' = y'/y$, where:

$$y = \sqrt[4]{\frac{x^2+1}{x^2-1}}$$

Word problems

6. § If a stone is thrown vertically upward from the surface of the moon with a velocity of 10 m/s, its heigh (in meters) after t seconds is $h = 10t - .83t^2$.
   (a) What is the velocity of the stone after 3 seconds?
   (b) What is the velocity of the stone after it has risen 25 meters?

7. § (a) Sodium chlorate crystals are easy to grow in the shape of cubes by allowing the solution of water and $NaClO_3$ to evaporate slowly. If V is the volume of such a cube with side length x, calculate dV/dx when $x = 3$ mm, and explain its meaning.
   (b) Show that the rate of change of the volume of such a cube with respect to its edge length (i.e. dV/dx) is equal to half the surface area of the cube. Explain geometrically why this is true.

8. § Boyle's Law states that when a sample of gas is compressed at constant temperature, the product of the pressure and the volume remains the same: $PV = C$.

   (a) Find the rate of change of volume with respect to pressure.

   (b) A sample of gas is in a container at low pressure and is compressed steadily (constant rate of increase in pressure) at constant temperature for 10 minutes. Is the volume decreasing more rapidly at the beginning or the end of the 10 minutes?

9. § Assume that in a particular reaction, one molecule of the product C is formed from one molecule of the reactant A and one molecule of the reactant B, and that the initial concentrations of A and B have the common value [A] = [B] = a moles per liter. Let x be the concentration of molecule C. Then

$$x = \frac{a^2kt}{akt + 1}$$

   for some constant k.

   (a) Explain how x relates to the concentrations of molecules A and B at time t.

   (b) Find the rate of reaction at time t.

   (c) Show that x satisfies the differential equation

$$\frac{dx}{dt} = k(a - x)^2$$

   Show that the right-hand side is proportional to the product of the concentrations at time t of molecules A and B.

   (d) What happens to the concentrations of A, B, and C as $t \to \infty$?

   (e) What happens to the rate of reaction as $t \to \infty$?

Questions on earlier material

10. What's the derivative of $\sin^2 x$? What's the derivative of $\cos^2 x$? What happens when you add them together and why?

11. Find numbers A and B so that

$$\frac{d}{dx}\left[Ae^x \cos x + Be^x \sin x\right] = e^x \cos x$$

12. Find numbers α and β so that $y = e^{\alpha x}\sin(\beta x)$ is a solution to the differential equation:

$$y'' + 4y' + 5y = 0$$

    Check that $y = e^{\alpha x}\cos(\beta x)$ is also a solution.

13. § Show that every curve in the family $y = ax^3$ is orthogonal to every curve in the family $x^2 + 3y^2 = b$, where a and b range over all real numbers.