Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Hyperbolic Functions

1. § Evaluate $\sinh(\ln 2)$ and $\sinh 2$.
2. § Prove that $\cosh x + \sinh x = e^x$ and that $\cosh x - \sinh x = e^{-x}$.
3. § Prove that $\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y$.
4. § Prove that $(\cosh x + \sinh x)^n = \cosh nx + \sinh nx$ for n any real number.
5. § If $\tanh x = 12/13$, find the values of the other hyperbolic functions of x.
6. Prove that $\lim_{x\to\infty} \tanh x = 1$, and that $\lim_{x\to-\infty} \tanh x = -1$.
7. Prove that $y = \tanh x$ if and only if $x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$. What is the domain of $x(y)$?
8. § Differentiate.
   (a) $x\sinh x - \cosh x$
   (b) $\ln\cosh x$
   (c) $e^{\cosh 3x}$
   (d) $\arctan(\tanh x)$
   (e) $(\operatorname{csch} t)(1 - \ln\operatorname{csch} t)$
   (f) $\sqrt[4]{\frac{1+\tanh x}{1-\tanh x}}$

Exponential Growth and Decay

9. § A bacteria culture initially contains 100 celss and grows at a rate proportional to its size. After an hour the population has increased to 420.
   (a) Find an expression for the number of bacteria after t hours.
   (b) Find the number of bacteria after 3 hours.
   (c) Find the growth rate after 3 hours.
   (d) When will the population reach 10,000 cells? What will the growth rate be then?
10. § Scientists can determine the age of ancient objects by the method of *radiocarbon dating*. The bombardment of the upper atmosphere by cosmic rays converts nitrogen to a radioactive isotope ^{14}C, with a half-life of about 5730 years. Vegetation absorbs carbon dioxide through the atmosphere and animal life assimilates ^{14}C through food chains. When a plant or animal dies, it stops replacing its carbon and the amount of ^{14}C begins to decrease through radioactive decay. Therefore the level of radioactivity must decal exponentially.

A parchment fragment was discovered that had about 75% as much ^{14}C radioactivity as does living plant material on Earth today. Estimate the age of the parchment.

11. § Experiments show that if the chemical reaction $N_2O_5 \to 2NO_2 + \frac{1}{2}O_2$ takes place at 45°C, the rate of reaction of dinitrogen pentoxide is proportional to its concentration as follows:

$$-\frac{d[N_2O_5]}{dt} = 0.0005[N_2O_5]$$

Assume the initial concentration of $[N_2O_5]$ is some number C.

(a) Find an expression for the concentration $[N_2O_5]$ after t seconds.

(b) How long will the reaction take to reduce the concentration of N_2O_5 to 90% of C?

12. § A freshly brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur? Assume Newton's Law of Cooling, which says that the rate of cooling is proportional to the temperature difference between the object and the ambient environment.

13. (a) In order to maintain the current standard of living, given the world economic structure, we have to have an annual growth rate in Gross World Product (GWP) of 3% per year. How long, at this rate, does it take for GWP to double? To increase by a factor of 10?

(b) GWP is roughly proportional to the amount of natural resources used. Let's say that we're currently using 1% capacity of the Earth's natural resources. How long until we use in one year the entire resources of the world?

(c) Let's say that because of sustained technological innovation, in order to increase the GWP by 3% per year, we need only increase our use of natural resources by a proportion of 1% per year. If we are currently 1% of the world's natural resources, how long until we run out? Hint: it's less than 100 years.

14. When banks loan money, they don't actually loan existing money. Rather, they create money to loan. They are limited by how much money they can create. The law says that for every dollar a bank has in one of its savings accounts, it is allowed to create another 90 cents to give out as a loan. (The dollar from the savings account is still there, and can still be spent by the person who owns the savings account.) This loan is then spent, and the recipient puts it into another bank, and that bank can now loan 90 cents times 0.9 = 81 cents.

(a) Continue the pattern $1, 1+.90, 1+.90+.81, \ldots$, for $n = 0, 1, 2, \ldots$, expressing how much money there is in the world if the original loan has been deposited and then a new loan made against it n times. Find a simple formula for the total amount of money that comes from \$1 after n deposit-loan iterations.

(b) Find the limit as $n \to \infty$ of the total amount of money coming from that one dollar.

(c) If each loan is kept for 1 year at 3% total interest, how much money will the banking industry make off of that original dollar?

(d) In fact, the government directly creates a certain amount of money, which it deposits in banks (via, e.g. the recent Bank Bailout, or by increasing spending without increasing taxes). Use part (b) to explain how much total money there should be in the economy for a given amount of government-created money. Use part (c) to explain how much profit the banks make for a given amount of government-created money.

(e) Where does the money the banks make come from?