Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Warm up

1. § Without a calculator, estimate the values of the following numbers (the first six are from last time; g–i are new). For each, specify if your estimate is high or low.
   (a) $(2.001)^5$
   (b) $e^{-0.015}$
   (c) $(8.06)^{2/3}$
   (d) $\tan 44°$
   (e) $\sqrt{99.8}$
   (f) $\sec 0.08$
   (g) $1000/1035$
   (h) $\arctan 1.01$
   (i) $\ln 0.99$

2. (From last time) § The circumference of a sphere is measured to be 84 cm, with a possible error of 0.5 cm. Use differentials to estimate the maximum error in the calculated surface area. What is the relative error? Use differentials to estimate the maximum error in the calculated volume. What is the relative error?

3. (From last time) § When blood flows along a blood vessel, Poiseuille's Law says that the flux F (the volume of blood per unit time that flows past a given point) is proportional to the fourth power of the radius R of the blood vessel:
$$F = kR^4$$
Show that the relative change in F is roughly four times the relative change in R. What happens if there is a clog that decreases the radius by 5%?

Geometry and Problem Solving

4. § Show that every curve in the family $y = ax^3$ is orthogonal to every curve in the family $x^2 + 3y^2 = b$, where a and b range over all real numbers.

5. § Show that the sum of the x- and y- intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y} = \sqrt{c}$ is equal to c.

6. § Let p and q be any two numbers. Show that the tangent lines to the parabola $y = ax^2+bx+c$ at the points with x-coordinates p and q must intersect at a point whose x-coordinate is halfway between p and q.

7. § The following fact follows from the Fundamental Theorem of Calculus: if two (continuously differentiable) functions have the same derivative and the same y-intercept, then they must be the same. Use this to prove:
$$\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$$

8. § If f is differentiable at a, where $a > 0$, evaluate the following limit in terms of a and $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

9. If y is a function of two variables, then dy depends on those variable and their differentials, so it depends on four numbers. Find dy when $y = r \sin \theta$.

10. In special relativity, the total energy E of an object depends on the the rest mass m, the momentum p, and the speed of light c by $E^2 = m^2c^4 + p^2c^2$. Prove that if p is much less than mc, then $E \approx mc^2 + \frac{1}{2}p^2/m$.

11. § Let $f(x) = (x-a)(x-b)(x-c)$ be an arbitrary cubic. Find the tangent line at $x = (a+b)/2$, and prove that it intersects the graph of $f(x)$ at $(c, 0)$.

12. § For which positive numbers a is it true that $a^x \geq 1 + x$ for all x?

13. § Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

14. (a) Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points in the plain, not all on the same line. Prove that there is exactly one circle $(x - h)^2 + (y - k)^2 = r^2$ that passes through all three points, and explain how to find h, k, and r.

    (b) Consider the parabola $y = 4x^2$. Prove that for any three tangent lines to the parabola, the circle defined by the three points of intersection of those three lines also passes through the point $(0, 1)$.