Math 1A: Discussion Exercises

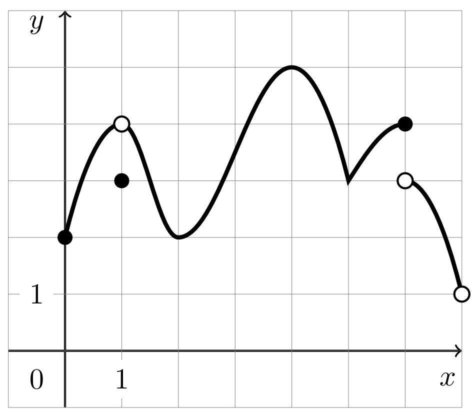
GSI: Theo Johnson-Freyd
http://math.berkeley.edu/~theojf/09Spring1A/

Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Maxima and minima

1. § Identify the absolute and local extrema of the following function:



2. § Sketch the graph of a function that is continuous on $[1, 5]$ with an absolute minimum at 1, an absolute maximum at 3, a local maximum at 2, and a local minimum at 4.

3. Sketch a function with a local minimum but no absolute minimum. Sketch a function with an absolute minimum but no local minimum.

4. § What are the absolute extrema of the function x^2 on the interval $0 < x < 2$? How about the function $\ln x$ on $0 < x \le 2$?

5. § Find the critical numbers of the following functions:
   (a) $x^3 + x^2 - x$ (b) $|3t - 4|$ (c) $x^{1/3} - x^{-2/3}$ (d) $4\theta - \tan\theta$

6. § Find how many critical numbers does the function $f(x)$ has given that

$$f'(x) = 5e^{-0.1|x|} \sin x - 1$$

7. § Find the absolute extrema of the function on the given interval:
   (a) $x^3 - 6x^2 + 9x + 2$, $[-1, 4]$ (c) $x/(x^2 + 1)$, $[0, 2]$ (e) $x + \cot(x/2)$, $[\pi/4, 7\pi/4]$
   (b) $(x^2 - 1)^3$, $[-1, 2]$ (d) $\sqrt[3]{x}\,(8 - x)$, $[0, 8]$ (f) $e^{-x} - e^{-2x}$, $[0, 1]$

8. § If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$ with $0 \le x \le 1$.

9. § Show that 5 is a critical number of the function $g(x) = 2 + (x - 5)^3$, but that g does not have a local extremum at 5.

10. § Prove that the function $f(x) = x^{101} + x^{51} + x + 1$ has no local extrema.

11. § A *cubic function* is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$.

    (a) Prove that a cubic function cannot have more than two critical numbers. Find examples showing that it can have zero, one, or two critical numbers.

    (b) What are the possible numbers of local extrema of a cubic function?

    (c) Use limits to prove that a cubic function has no absolute extrema.

Geometry and Problem Solving

12. § Let p and q be any two numbers. Show that the tangent lines to the parabola $y = ax^2 + bx + c$ at the points with x-coordinates p and q must intersect at a point whose x-coordinate is halfway between p and q.

13. § The following fact follows from the Fundamental Theorem of Calculus: if two (continuously differentiable) functions have the same derivative and the same y-intercept, then they must be the same. Use this to prove:

$$\sin^{-1}(\tanh x) = \tan^{-1}(\sinh x)$$

14. § If f is differentiable at a, where $a > 0$, evaluate the following limit in terms of a and $f'(a)$:

$$\lim_{x \to a} \frac{f(x) - f(a)}{\sqrt{x} - \sqrt{a}}$$

15. If y is a function of two variables, then dy depends on those variable and their differentials, so it depends on four numbers. Find dy when $y = r \sin\theta$.

16. In special relativity, the total energy E of an object depends on the the rest mass m, the momentum p, and the speed of light c by $E^2 = m^2c^4 + p^2c^2$. Prove that if p is much less than mc, then $E \approx mc^2 + \frac{1}{2}p^2/m$.

17. § Let $f(x) = (x-a)(x-b)(x-c)$ be an arbitrary cubic. Find the tangent line at $x = (a+b)/2$, and prove that it intersects the graph of $f(x)$ at $(c, 0)$.

18. § For which positive numbers a is it true that $a^x \geq 1 + x$ for all x?

19. § Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

20. (a) Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points in the plain, not all on the same line. Prove that there is exactly one circle $(x - h)^2 + (y - k)^2 = r^2$ that passes through all three points, and explain how to find h, k, and r.

    (b) Consider the parabola $y = 4x^2$. Prove that for any three tangent lines to the parabola, the circle defined by the three points of intersection of those three lines also passes through the point $(0, 1)$.