Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Mean Value Theorem

1. § Let $f(x) = 1 - x^{2/3}$. Show that $f(-1) = f(1)$ but that there is no number c in $(-1, 1)$ such that $f'(x) = 0$. Why doesn't this contradict Rolle's Theorem?

2. § Let $f(x) = (x - 3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4 - 1)$. Why doesn't this contradict the Mean Value Theorem?

3. § Verify that the function satisfies the hypotheses (the "if" part) of MVT on the given interval. Then find all numbers c that satisfy the conclusion of MVT.
   (a) $3x^2 + 2x + 5$, $[-1, 1]$
   (b) e^{-2x}, $[0, 3]$

4. § Use Rolle's Theorem to show that if f is differentiable on $\mathbb{R}$ and has at least two roots, then f' has at least one root. Show that if f is twice differentiable and has at least three roots, then f'' has at least one root. Generalize.

5. § Show that the equation $1 + 2x + x^3 + 4x^5 = 0$ has exactly one real root.

6. § How many real roots can the equation $x^4 + 4x + c = 0$ have?

7. Let f be a polynomial of degree n. Show that if all of the roots of f are real (recall that a polynomial of degree n has exactly n roots, if you count complex roots too), then all the roots of all derivatives of f are real.

8. (a) Use the Mean Value Theorem to prove the *Footrace Theorem*: if f and g are two differentiable functions such that $f(0) = g(0)$ and for every x, $f'(x) = g'(x)$, then $f(x) = g(x)$ for every x.
   (b) Explain the meaning of the Footrace Theorem in the case when x is time and f and g are the distances traveled by Felicia and Gabriel in a sprint.
   (c) Use the Footrace Theorem to prove:
   $$\arcsin \tanh x = \arctan \sinh x$$

9. Use the Footrace Theorem to prove that:
   $$\text{If } f'(x) = kf(x) \text{ for all } x, \text{ then } f(x) = f(0)\, e^{kx}.$$
   You may want to use the following outline:

(a) Show that the constant function $f(x) = 0$ is not a counterexample to the theorem.

(b) Show that if $f(x)$ satisfies the "if" part of what you're trying to prove, then for any number a, $g(x) = f(x - a)$ also satisfies the "if" part. Conversely, show that if $g(x)$ satisfies the "then" part, then so does $f(x) = g(x + a)$.

(c) Conclude that if there is a counterexample to the theorem, then there is a counterexample with $f(0) \neq 0$.

(d) Assume that f satisfies the conditions of the statement to be proven, and that $f(0) \neq 0$. Consider $F(x) = \ln\left|f(x)/f(0)\right|$, and find $F'(x)$.

(e) What's another function with the same derivative as F? Hence, apply the Footrace theorem.

(f) Solve for $f(x)$.

10. § Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f(a) = g(a)$ and that $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. Hint: Apply MVT to $h = f - g$.

11. § A number a is a *fixed point* of a function f if $f(a) = a$. Prove that if f is differentiable on $\mathbb{R}$ and $f'(x) \neq 1$ for all x, then f has at most one fixed point.

12. (a) § Show that if $f'(x) = c$, where c is a constant, then $f(x) = cx + d$, for some constant d.

    (b) Show that if $f'(x) = bx + c$, where b and c are constants, then $f(x) = bx^2/2 + cx + d$, for some constant d.

    (c) Show that if f' is a polynomial, then so is f.

Hard exercises from earlier

13. § A *cubic function* is a polynomial of degree 3; that is, it has the form $f(x) = ax^3 + bx^2 + cx + d$ with $a \neq 0$.

    (a) Prove that a cubic function cannot have more than two critical numbers. Find examples showing that it can have zero, one, or two critical numbers.

    (b) What are the possible numbers of local extrema of a cubic function?

    (c) Use limits to prove that a cubic function has no absolute extrema.

14. In special relativity, the total energy E of an object depends on the the rest mass m, the momentum p, and the speed of light c by $E^2 = m^2c^4 + p^2c^2$. Prove that if p is much less than mc, then $E \approx mc^2 + \frac{1}{2}p^2/m$.

15. § For which positive numbers a is it true that $a^x \geq 1 + x$ for all x?

16. § Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

17. (a) Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be three points in the plane, not all on the same line. Prove that there is exactly one circle $(x - h)^2 + (y - k)^2 = r^2$ that passes through all three points, and explain how to find h, k, and r.

    (b) Consider the parabola $y = 4x^2$. Prove that for any three tangent lines to the parabola, the circle defined by the three points of intersection of those three lines also passes through the point $(0, 1)$.