Math 1A: Discussion Exercises

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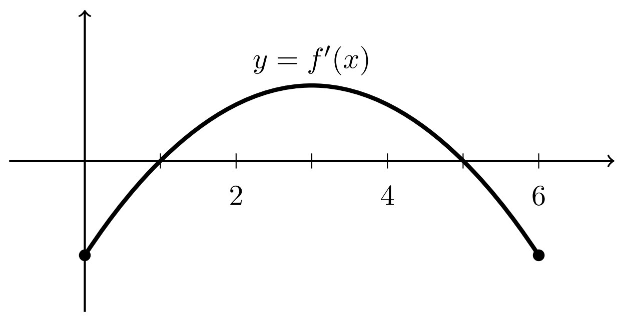
Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Derivatives and graphing

1. § To the right, the graph of the *derivative* f' of a function f is shown. On what intervals is f increasing? Decreasing? Where is f concave up? Concave down? At what values does f have a local maximum or minimum? Inflection points?



2. § Sketch the graph of a function f such that $f'(1) = f'(-1) = 0$, $f'(x) < 0$ if $|x| < 1$, $f'(x) > 0$ if $1 < |x| < 2$, $f'(x) = -1$ if $|x| < 2$, $f''(x) < 0$ if $-2 < x < 0$, and such that $(0, 1)$ is an inflection point of $y = f(x)$.

3. § Suppose that $f(x) = 2$, $f'(3) = \frac{1}{2}$, and $f'(x) > 0$ and $f''(x) < 0$ for all x.
   (a) Sketch a possible graph of f.
   (b) How many solutions does the equation $f(x) = 0$ have?
   (c) Is it possible that $f'(2) = 13$?

4. § For each of the following functions, find: intervals when f is increasing; intervals when f is decreasing; intervals when f is concave up; intervals when f is concave down; local extreme of f; inflection points. Then sketch a graph of the function.
   (a) $f(x) = 4x^3 + 3x^2 - 6x + 1$
   (b) $f(x) = x^2 \ln x$
   (c) $f(x) = \sqrt{x}e^{-x}$
   (d) $f(x) = 200 + 8x^3 + x^4$
   (c) $f(x) = x^{1/3}(x + 4)$
   (d) $f(x) = x + \cos x$

5. Sketch a careful graph of $y = (x^2 + x + 1)e^x$. Label any interesting features (intercepts, asymptotes, extrema, points of inflection).

6. (a) Let $f(x) = (x - r)e^x$. Use calculus to sketch a graph of $f(x)$, and label the zeros, local extrema, and inflection points. Also label the y-intercept and any horizontal asymptotes.
   (b) Let $f(x) = (x^2 + bx + c)e^x$. What is the behavior of $f(x)$ as $x \to \pm\infty$? Use the Mean Value Theorem to show that if $f(x)$ has one or two zeros, then it must have two local extrema.

(c) More generally, let $f(x) = p(x)e^x$, where $p(x)$ is a polynomial of degree n. Show that if $f(x)$ has exactly n (real, distinct) zeros, then it also has exactly n local extrema and exactly n inflection points.

(d) **(Harder)** Let's return to the case when $f(x) = (x^2 + bx + c)e^x$. Prove that the zeros of f correspond to the zeros of $q_0(x) = x^2 + bx + c = f(x)/e^x$, and the number of these is classified by the determinant $b^2 - 4c$.

For n a non-negative integer, define $q_n(x)$ to be $f^{(n)}(x)/e^x$, the polynomial part of the nth derivative of e^x. Prove that $q_n(x)$ is a quadratic for any n.

How if the determinant of $q_1(x)$ related to the determinant of $q_0(x)$? Is this the same as the relationship between the determinants of $q_n(x)$ and $q_{n+1}(x)$ for arbitrary n? Why or why not?

Prove that for n large enough, $q_n(x)$, and hence $f^{(n)}(x)$, will have two roots.

7. § Suppose the derivative of a function f is $f'(x) = (x+1)^2(x-3)^5(x-6)^4$. On what intervals is f increasing? What are the local maxima of f?

8. § Use calculus to sketch the family of curves $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant.

9. § Find the value of x such that $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ increases most rapidly.

10. § Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

11. § For what values of the numbers a and b does the function $f(x) = axe^{bx^2}$ have the maximum value $f(2) = 1$?

12. § Show that the curve $y = \dfrac{1+x}{1+x^2}$ has three points of inflection and that they all lie on one straight line.

13. § Show that the curves $y = e^{-x}$ and $y = -e^{-x}$ touch the curve $y = e^{-x}\sin x$ at its inflection points.

14. § Show that $\tan x > x$ for $0 < x < \pi/2$. Hint: show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.

15. § Show that a cubic function always has precisely one point of inflection. Show that if the graph has three x-intercepts x_1, x_2, and x_3, then the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/2$.

    What is the similar statement about local extrema of a quadratic function?

16. (a) § Show that $f(x) = x^4$ is such that $f''(0) = 0$ but $(0, 0)$ is not an inflection point of the graph of f.

    (b) § Show that $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

    (c) § Let f be any function. Use the First Derivative Test and Fermat's Theorem on the function $g = f'$ to show that if $(c, f(c))$ is an inflection point and f'' exists in an open interval that contains c, then $f''(c) = 0$.