Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

L'Hospital's Rule

1. § Given that
$$\lim_{x\to a} f(x) = 0, \quad \lim_{x\to a} g(x) = 0, \quad \lim_{x\to a} h(x) = 1, \quad \lim_{x\to a} p(x) = \infty, \quad \lim_{x\to a} q(x) = \infty,$$
determine which of the following are indeterminate, and evaluate the limit of those that are not indeterminate.
$$\lim_{x\to a} \frac{f(x)}{g(x)}, \quad \lim_{x\to a} \frac{f(x)}{p(x)}, \quad \lim_{x\to a} \frac{p(x)}{h(x)}, \quad \lim_{x\to a} \frac{p(x)}{q(x)},$$
$$\lim_{x\to a} \big[f(x)\,p(x)\big], \quad \lim_{x\to a} \big[h(x)\,p(x)\big], \quad \lim_{x\to a} \big[q(x)\,p(x)\big],$$
$$\lim_{x\to a} \big[f(x) - p(x)\big], \quad \lim_{x\to a} \big[q(x) - p(x)\big], \quad \lim_{x\to a} \big[q(x) + p(x)\big],$$
$$\lim_{x\to a} \big[f(x)\big]^{g(x)}, \quad \lim_{x\to a} \big[f(x)\big]^{p(x)}, \quad \lim_{x\to a} \big[h(x)\big]^{p(x)},$$
$$\lim_{x\to a} \big[p(x)\big]^{g(x)}, \quad \lim_{x\to a} \big[p(x)\big]^{q(x)}, \quad \lim_{x\to a} \sqrt[q(x)]{p(x)},$$

2. § Find the limit. Use l'Hospital's Rule where appropriate, but also look for other methods.
   (a) $\displaystyle\lim_{x\to 1} \frac{x^9 - 1}{x^5 - 1}$
   (b) $\displaystyle\lim_{x\to 1} \frac{x^a - 1}{x^b - 1}$
   (c) $\displaystyle\lim_{x\to 0} \frac{\tan px}{\tan qx}$
   (d) $\displaystyle\lim_{\theta\to \pi/2} \frac{1 - \sin\theta}{\csc\theta}$
   (e) $\displaystyle\lim_{x\to \infty} \frac{\ln x}{\sqrt{x}}$
   (f) $\displaystyle\lim_{x\to \infty} \frac{x + x^2}{1 - 2x^2}$

3. **More MVT:** This is a follow-up to today's quiz. In particular, on the quiz you should have found exactly two extrema and exactly two inflection points.
   (a) Let $f(x) = (x - r)e^x$. Use calculus to sketch a graph of $f(x)$, and label the zeros, local extrema, and inflection points. Also label the y-intercept and any horizontal asymptotes.
   (b) Let $f(x) = (x^2 + bx + c)e^x$. What is the behavior of $f(x)$ as $x \to \pm\infty$? Use the Mean Value Theorem to show that if $f(x)$ has one or two zeros, then it must have two local extrema.
   (c) More generally, let $f(x) = p(x)e^x$, where $p(x)$ is a polynomial of degree n. Show that if $f(x)$ has exactly n (real, distinct) zeros, then it also has exactly n local extrema and exactly n inflection points. Hint: Show that $f'(x)/e^x$ is a polynomial of degree n, so does not have more than n roots. Then use MVT to show that it has at least n roots.

   For a harder fourth part, look on Monday's handout or ask Theo.