Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Graphing

1. § Sketch the following curves:
   (a) $y = x^3 + 6x^2 + 9x$
   (b) $y = \dfrac{x}{x^3 - 1}$
   (c) $y = 2\sqrt{x} - x$
   (d) $y = \dfrac{\sin x}{2 + \cos x}$
   (e) $y = \ln(x^2 - 3x + 2)$
   (f) $y = \arctan\left(\dfrac{x-1}{x+1}\right)$

2. § Sketch the curve, and find the slant asymptote:
   (a) $y = \dfrac{x^2 + 12}{x - 2}$
   (b) $y = \dfrac{(x+1)^3}{(x-1)^2}$
   (c) $xy = x^2 + 4$

3. § Show that each of the following curves has two slant asymptotes, find both of them, and sketch their graphs:
   (a) $y = x - \arctan x$
   (b) $y = \sqrt{x^2 + 4x}$
   (c) $y = \sqrt{x^2 + 2x + 2} - 2x$

4. (a) § Sketch the curve $y = e^{-x} \sin x$.
   (b) § Show that the curves $y = e^{-x}$ and $y = -e^{-x}$ touch the curve $y = e^{-x} \sin x$ at its inflection points.

5. § In the theory of relativity, the mass of a particle is
$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}}$$
where m_0 is the rest mass of the particle, m is the mass when the particle moves with speed v, and c is the speed of light. Sketch the graph of m as a function of v. What happens in the limit as $v \to c$?

6. § In the theory of relativity, the energy of a particle is
$$E = \sqrt{m_0^2 c^4 + h^2 c^2/\lambda^2}$$
where m_0 is the rest mass of the particle, λ is its wave length, and h is Planck's constant. Sketch the graph of E as a function of λ. What are the asymptotics as $\lambda \to \infty$? What about as $\lambda \to 0$?

Hard Problems from Previous Days

7. § Suppose the derivative of a function f is $f'(x) = (x+1)^2(x-3)^5(x-6)^4$. On what intervals is f increasing? What are the local maxima of f?

8. § Use calculus to sketch the family of curves $y = x^3 - 3a^2x + 2a^3$, where a is a positive constant.

9. § Find the value of x such that $f(x) = \dfrac{x+1}{\sqrt{x^2+1}}$ increases most rapidly.

10. § Find a cubic function $f(x) = ax^3 + bx^2 + cx + d$ that has a local maximum value of 3 at $x = -2$ and a local minimum value of 0 at $x = 1$.

11. § For what values of the numbers a and b does the function $f(x) = axe^{bx^2}$ have the maximum value $f(2) = 1$?

12. § Show that the curve $y = \dfrac{1+x}{1+x^2}$ has three points of inflection and that they all lie on one straight line.

13. § Show that $\tan x > x$ for $0 < x < \pi/2$. Hint: show that $f(x) = \tan x - x$ is increasing on $(0, \pi/2)$.

14. § Show that a cubic function always has precisely one point of inflection. Show that if the graph has three x-intercepts x_1, x_2, and x_3, then the x-coordinate of the inflection point is $(x_1 + x_2 + x_3)/2$.

    What is the similar statement about local extrema of a quadratic function?

15. (a) § Show that $f(x) = x^4$ is such that $f''(0) = 0$ but $(0, 0)$ is not an inflection point of the graph of f.
    (b) § Show that $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.
    (c) § Let f be any function. Use the First Derivative Test and Fermat's Theorem on the function $g = f'$ to show that if $(c, f(c))$ is an inflection point and f'' exists in an open interval that contains c, then $f''(c) = 0$.