Math 1A: Discussion Exercises

GSI: Theo Johnson-Freyd
http://math.berkeley.edu/∼theojf/09Spring1A/

Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Optimization Problems

1. (a) § Find two positive numbers whose product is 100 and whose sum is a minimum.
   (b) Find two positive numbers whose product is 100 and whose sum is a maximum.

2. § A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soul (measured in the appropriate units) is:
$$Y = \frac{kN}{1+N^2}$$
where k is a positive constant. What nitrogen level gives the best yield?

3. § Consider the following problem: A farmer with 750 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What is the largest possible total area of the four pens?
   (a) Draw several diagrams illustrating the situations, some with shallow, wide pens and some with deep, narrow pens. Find the total areas of these configurations. Does it appear that there is a maximum area? If so, estimate it.
   (b) Draw a diagram illustrating the general situation. Introduce notation and label the diagram with symbols.
   (c) Write an expression for the total area.
   (d) Use the given information to write an equation that relates the variables.
   (e) Use part (d) to write the total area as a function of one variable.
   (f) Finish solving the problem and compare the answer with your estimate in part (a).

4. § A farmer wants to fence an area of 1.5 million square feet in a rectangular field and then divide it in half with a fence parallel to one of the sides of the rectangle. How can she do this so as to minimize the cost of the fence?

5. (a) § Show that of all the rectangles with a given area, the one with the smallest perimeter is a square.
   (b) § Show that of all the rectangles with a given perimeter, the one with the greatest area is a square.

6. § Find the point on the line $y = 4x + 7$ that is closest to the origin.

7. § Find the points on the ellips $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

8. § Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. You may assume that the sides of the rectangle are parallel to the axes.

9. (a) § Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two of the sides of the rectangle lie along with legs.

   (b) Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if one of the sides of the rectangle lies along the hypotenuse.

10. § Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

11. § A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

12. § A cylindrical can without a top is made to contain $V\,\mathrm{cm}^3$ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

13. § A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

14. § A cone-shaped paper drinking cup is to be made to hold $27\,\mathrm{cm}^3$ of water. Find the height and radius of the cup that will use the smallest amount of paper.

15. § An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is

$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$

    where μ is a constant called the *coefficient of friction*. For what value of θ is F smallest?

16. § If $C(x)$ is the cost of producing x units of a commodity, then the *average cost* per unit is $C(x)/x$, and the *marginal cost* per unit is $C'(x)$. Prove that if the average cost is a minimum, then the average cost equals the marginal cost.

17. § If $P = (a, a^2)$ is any point on the parabola $y = x^2$, except for the origin, let Q by the point where the normal line intersects the parabola gain. Show that the line segment PQ has the shortest possible length when $a = \pm 1/\sqrt{2}$.

18. § Let $A = (a, a^2)$ and $B = (b, b^2)$ be two fixed points on the parabola $y = x^2$, with $a \leq b$. Find the point $P = (x, x^2)$ on the arc between A and B (i.e. $a \leq x \leq b$) so that the triangle APB has the largest possible area.