Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Optimization Problems

1. § Find the points on the ellips $4x^2 + y^2 = 4$ that are farthest away from the point $(1, 0)$.

2. § Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$. You may assume that the sides of the rectangle are parallel to the axes.

3. (a) § Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if two of the sides of the rectangle lie along with legs.
   (b) Find the area of the largest rectangle that can be inscribed in a right triangle with legs of lengths 3 cm and 4 cm if one of the sides of the rectangle lies along the hypotenuse.

4. § Find the dimensions of the isosceles triangle of largest area that can be inscribed in a circle of radius r.

5. § A right circular cylinder is inscribed in a cone with height h and base radius r. Find the largest possible volume of such a cylinder.

6. § A cylindrical can without a top is made to contain $V\,\mathrm{cm}^3$ of liquid. Find the dimensions that will minimize the cost of the metal to make the can.

7. § A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

8. § A cone-shaped paper drinking cup is to be made to hold $27\,\mathrm{cm}^3$ of water. Find the height and radius of the cup that will use the smallest amount of paper.

9. § An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is
$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$
where μ is a constant called the *coefficient of friction*. For what value of θ is F smallest?

10. § If $C(x)$ is the cost of producing x units of a commodity, then the *average cost* per unit is $C(x)/x$, and the *marginal cost* per unit is $C'(x)$. Prove that if the average cost is a minimum, then the average cost equals the marginal cost.

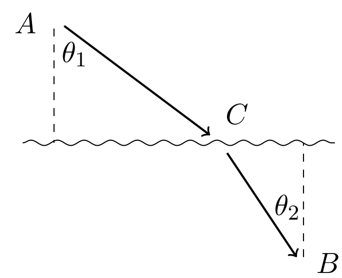
11. § If $P = (a, a^2)$ is any point on the parabola $y = x^2$, except for the origin, let Q by the point where the normal line intersects the parabola gain. Show that the line segment PQ has the shortest possible length when $a = \pm 1/\sqrt{2}$.

12. § Let $A = (a, a^2)$ and $B = (b, b^2)$ be two fixed points on the parabola $y = x^2$, with $a \leq b$. Find the point $P = (x, x^2)$ on the arc between A and B (i.e. $a \leq x \leq b$) so that the triangle APB has the largest possible area.

13. § Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to *Fermat's Principle*, a ray of light will travel between a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that

$$\frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2}$$

where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as *Snell's Law.*



14. (a) Let C be a fixed positive number. Prove that the minimum sum of two positive numbers whose product is C occurs when the two numbers are equal.

    (b) Prove that the minimum sum of *three* positive numbers whose product is C occurs when the three numbers are equal. Hint: Call the numbers x, y, and z. Then the product of y and z is C/x; if we pretend that x is fixed, then how we can minimum the sum of the other two? What is this minimum sum, as a function of x? So what is the minimum possibility for x plus this sum?

    (c) Generalize to the sum of n positive numbers with a fixed product.