Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

Optimization Problems: Applications to Business and Economics

1. § If $C(x)$ is the cost of producing x units of a commodity, then the *average cost* per unit is $C(x)/x$, and the *marginal cost* per unit is $C'(x)$. Prove that if the average cost is a minimum, then the average cost equals the marginal cost.

2. § Let $R(x)$ be the revenue brought in by selling x units of a commodity, and $C(x)$ the cost of producing those x units. Then the *profit* from selling x units is $P(x) = R(x) - C(x)$. Define the *marginal cost* per unit to be $C'(x)$ and the *marginal revenue* per unit to by $R'(x)$. Prove that if the company sells a number of units so as to maximize its profits, then the marginal cost is equal to the marginal revenue.

3. If a company charges more for a commodity, then the number of units of the commodity the company can sell generally decreases. Define the *price function* $p(x)$ to be the unit price for which the company will sell x many units of the commodity. Why is $p(x)$ a decreasing function? Why does $R(x) = x\,p(x)$?

   (a) Explain how the marginal revenue per unit depends on the demand. In particular, show that when $x = 0$, the marginal revenue per unit is exactly the price.

   (b) If $p(x) = b - mx$, where $m, b > 0$, for what value of x is the marginal revenue 0? For what values is it positive? For what values is it negative?

   (c) If $p(x) = Ce^{-kx}$ for $C, k > 0$, does there ever come a time when the marginal revenue is negative? If so, when? What about for $p(x) = ae^{-kx} + b$ where $a, b, k > 0$?

4. Economists are not known for their mathematics skills. Thus, they generally assume that $p(x)$ and $C(x)$ are linear functions. Let's say that $p(x) = b - mx$ and $C(x) = a + lx$, where $b, m, a, l > 0$. In terms of a, b, m, l, how many units of a commodity should the company produce, and what price should the company charge, so as to maximize their profits? What is the maximum profit the company can make?

5. In your group, discuss how this shows that sometimes a company can increase its profits by raising its prices, and sometimes it can do so by lowering its prices. Explain how the company can determine whether it should raise or lower its prices in terms of the marginal cost and marginal revenue.

6. In your group, discuss the problems with these models. For example, if $p(x)$ is linear, then for large enough x, $p(x)$ will be negative. Is this reasonable? If $C(x)$ is linear, then the marginal

cost is constant. Is this reasonable, even as $x \to \infty$? What happens to a company that raises its prices if there is a competing company making a similar product?

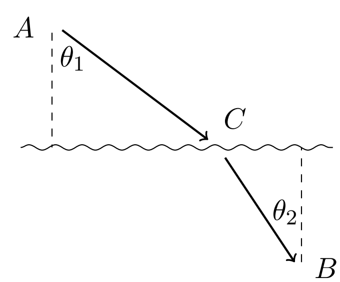
Other Optimization Problems

7. § A fence 8 ft tall runs parallel to a tall building at a distance of 4 ft from the building. What is the length of the shortest ladder that will reach from the ground over the fence to the wall of the building?

8. § An object with weight W is dragged along a horizontal plane by a force acting along a rope attached to the object. If the rope makes an angle θ with the plane, then the magnitude of the force is
$$F = \frac{\mu W}{\mu \sin\theta + \cos\theta}$$
where μ is a constant called the *coefficient of friction*. For what value of θ is F smallest?

9. § Let $A = (a, a^2)$ and $B = (b, b^2)$ be two fixed points on the parabola $y = x^2$, with $a \le b$. Find the point $P = (x, x^2)$ on the arc between A and B (i.e. $a \le x \le b$) so that the triangle APB has the largest possible area.

10. § Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to *Fermat's Principle*, a ray of light will travel between a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as *Snell's Law*.



11. (a) Let C be a fixed positive number. Prove that the minimum sum of two positive numbers whose product is C occurs when the two numbers are equal.

    (b) Prove that the minimum sum of *three* positive numbers whose product is C occurs when the three numbers are equal. Hint: Call the numbers x, y, and z. Then the product of y and z is C/x; if we pretend that x is fixed, then how we can minimum the sum of the other two? What is this minimum sum, as a function of x? So what is the minimum possibility for x plus this sum?

    (c) Generalize to the sum of n positive numbers with a fixed product.