Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

The Definite Integral

1. § Using only the following facts, and the linearity (Sum, Difference, and Constant Multiple rules) of $\sum$ and $\int$, evaluate the integrals below. The facts:

$$\sum_{i=1}^{n} 1 = \frac{n}{1}, \quad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+\frac{1}{2})(n+1)}{3}, \quad \sum_{n=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

The integrals:

$$\text{(a)} \int_{-1}^{5} (1+3x)\,dx \qquad \text{(b)} \int_{0}^{2} (2-x^2)\,dx \qquad \text{(c)} \int_{0}^{5} (1+2x^2)\,dx \qquad \text{(d)} \int_{1}^{2} x^3\,dx$$

2. Using the facts in the previous problem, prove the following formulas:

$$\text{(a)} \int_a^b x\,dx = \frac{b^2-a^2}{2} \qquad \text{(b)} \int_a^b x^2\,dx = \frac{b^3-a^3}{3} \qquad \text{(c)} \int_a^b x^3\,dx = \frac{b^4-a^4}{4}$$

3. Based on the formulas from the previous problem, guess the formula for $\int_a^b x^n\,dx$. Check that your formula works for $n=0$. What happens when $n=-1$? For which n are you confident that your formula is correct?

4. § Evaluate each of the following integrals by interpreting them in terms of areas.

$$\text{(a)} \int_0^3 \left(\frac{1}{2}x - 1\right) dx \qquad \text{(c)} \int_{-2}^{2} \sqrt{4-x^2}\,dx \qquad \text{(e)} \int_{-1}^{2} |x|\,dx$$

$$\text{(b)} \int_{-1}^{3} (3-2x)\,dx \qquad \text{(d)} \int_{-3}^{0} \left(1+\sqrt{9-x^2}\right) dx \qquad \text{(f)} \int_{-1}^{1} x^5 \cos x\,dx$$

5. (a) Use the fact that $\cos x = \sin(\frac{\pi}{4} - x)$ to prove that $\int_0^{\pi/4} \cos^2 x\,dx = \int_0^{\pi/4} \sin^2 x\,dx$.
   (b) Use the previous problem and the Pythagorean theorem to find $\int_0^{\pi/4} \sin^2 x$.

6. § If $\int_1^5 f(x)\,dx = 12$ and $\int_4^5 f(x)\,dx = 3.6$, find $\int_1^4 f(x)\,dx$.

7. § Find $\int_0^5 f(x)\,dx$ if $f(x) = \begin{cases} 3, & x < 3 \\ x, & x \geq 3 \end{cases}$.

8. § Recall that if $f(x) \le g(x)$ for every $x \in [a,b]$, then $\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$. Use this fact to prove that:

$$2 \le \int_{-1}^{1} \sqrt{1+x^2}\,dx \le 2\sqrt{2}$$

9. § By finding the maximum and minimum values of each of the following functions on the corresponding interval, bound the value of the integral:

(a) $\displaystyle\int_1^4 \sqrt{x}\,dx$ (b) $\displaystyle\int_0^2 \frac{1}{1+x^2}\,dx$ (c) $\displaystyle\int_{\pi/4}^{\pi/3} \tan x\,dx$ (d) $\displaystyle\int_0^2 xe^{-x}\,dx$

10. § Prove that $\displaystyle\int_1^3 \sqrt{x^4+1}\,dx \ge \frac{26}{3}$.

11. § Prove that if f is continuous on $[a,b]$, then

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b \big|f(x)\big|\,dx$$

12. § Use the previous problem to prove for any continuous function $f(x)$:

$$\left|\int_0^{2\pi} f(x)\sin x\,dx\right| \le \int_0^{2\pi} \big|f(x)\big|\,dx$$

13. § By choosing $x^* = \sqrt{x_{i-1}x_i}$, evaluate $\displaystyle\int_1^2 x^{-2}\,dx$. Hint: $\frac{1}{m(m+1)} = \frac{1}{m} - \frac{1}{m+1}$.