Math 1A: Discussion Exercises

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Find two or three classmates and a few feet of chalkboard. As a group, try your hand at the following exercises. Be sure to discuss how to solve the exercises — *how* you get the solution is much more important than *whether* you get the solution. If as a group you agree that you all understand a certain type of exercise, move on to later problems. You are not expected to solve all the exercises: in particular, the last few exercises may be very hard.

Many of the exercises are from *Single Variable Calculus: Early Transcendentals for UC Berkeley* by James Stewart; these are marked with an §. Others are my own, or are independently marked.

u-substitutions

1. § Evaluate the indefinite integral:

(a) $\displaystyle\int \frac{(\ln x)^2}{x}\,dx$

(b) $\displaystyle\int \frac{dx}{ax+b}$, $a \neq 0$

(c) $\displaystyle\int \frac{\cos\sqrt{t}}{\sqrt{t}}\,dt$

(d) $\displaystyle\int \cos\theta \sin^6\theta\,d\theta$

(e) $\displaystyle\int (1+\tan\theta)^5 \sec^2\theta\,d\theta$

(f) $\displaystyle\int e^x\sqrt{1+e^x}$

(g) $\displaystyle\int \frac{z^2\,dz}{\sqrt[3]{1+z^3}}$

(h) $\displaystyle\int \frac{\cos(\pi/x)}{x^2}\,dx$

(i) $\displaystyle\int \cot x\,dx$

(j) $\displaystyle\int \frac{\sin x}{1+\cos^2 x}\,dx$

(k) $\displaystyle\int \frac{1+x}{1+x^2}\,dx$

(l) $\displaystyle\int \frac{x\,dx}{\sqrt[4]{x+2}}$

2. § Evaluate the definite integral:

(a) $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{x^2\sin x}{1+x^6}\,dx$

(c) $\displaystyle\int_0^{13} \frac{dx}{\sqrt[3]{(1+2x)^2}}$

(d) $\displaystyle\int_0^a x\sqrt{x^2+a^2}\,dx$, $a > 0$

(e) $\displaystyle\int_0^a x\sqrt{a^2-x^2}\,dx$

(f) $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$

(g) $\displaystyle\int_e^{e^4} \frac{dx}{x\sqrt{\ln x}}$

(h) $\displaystyle\int_0^{1/2} \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$

(i) $\displaystyle\int_0^1 \frac{e^z+1}{e^z+z}\,dz$

(j) $\displaystyle\int_0^{T/2} \sin(2\pi t/T - \alpha)\,dt$

3. § Evaluate $\int_0^1 x\sqrt{1-x^4}\,dx$ by making a substitution and interpreting the resulting integral in terms of an area.

4. Evaluate $\int \sec^2\theta \tan\theta\, d\theta$ in three different ways:

   (a) by making the substitution $u = \tan\theta$

   (b) by making the substitution $u = \sec\theta$

   (c) by making the substitution $u = \cos\theta$

5. What's wrong with the following:

$$\int \frac{1}{2x}\,dx = \frac{1}{2}\int x^{-1}\,dx = \frac{1}{2}\ln x.$$

However, $\displaystyle\int \frac{1}{2x}\,dx = \int \frac{1}{u}\frac{du}{2} = \frac{1}{2}\int u^{-1}\,du = \frac{1}{2}\ln u = \frac{1}{2}\ln(2x).$

Therefore $\displaystyle\frac{1}{2}\ln x = \frac{1}{2}\ln(2x)$, so $\ln x = \ln(2x)$, and therefore $x = 2x$ for any x.

6. § If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$. Also find $\int_0^2 x f(x^2)\,dx$.

7. § If a and b are positive numbers, prove that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

8. (a) § If f is continuous on $[0,\pi]$, use the substitution $u = \pi - x$ to prove

$$\int_0^\pi x\, f(\sin x)\,dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

   (b) § Find $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx.$

9. § Suppose that the coefficients of the cubic polynomial $p(x) = a + bx + cx^2 + dx^3$ satisfy $0 = a + \frac{b}{2} + \frac{c}{3} + \frac{d}{4}$. Prove that the equation $p(x) = 0$ has a root between 0 and 1. *Hint:* What is $\int_0^1 p(x)\,dx$?

   Generalize to an nth-degree polynomial.

10. (a) § For any number c, let $f_c(x)$ be the smaller of the two numbers $(x-c)^2$ and $(x-c-2)^2$. Find a formula for $\int_0^1 f_c(x)\,dx$ that only depends on c. *Hint:* draw a picture. *Second hint:* there are lots of piece-wise functions going on.

    (b) § Let $f_c(x)$ be as in the previous question, and define the function $g(c) = \int_0^1 f_c(x)\,dx$. Find the maximum and minimum values of $g(c)$ for $-2 \le c \le 2$.