Math 1A: Quiz 6 *ANSWERS*

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You must always justify your answers. This means: show your work, show it neatly, and when in doubt, use words (and pictures!) to explain your reasoning. No justification = no points.

1. (5 pts) Find y' in terms of x and y:

$$\ln(x+y) = \arctan(\sinh x) + xy^2$$

Perform any obvious simplifications — remember the hyperbolic Pythagorean formula — but don't be worried by a messy answer.

We use implicit differentiation, recalling the Chain and Product rules:

$$\begin{aligned}
\big(\ln(x+y)\big)' &= \big(\arctan(\sinh x) + xy^2\big)' \\
\frac{1+y'}{x+y} &= \arctan'(\sinh x)\sinh' x + y^2 + x(y^2)' \\
&= \frac{1}{\sinh^2 x + 1}\sinh' x + y^2 + 2xyy' \\
\frac{1}{x+y} + \frac{1}{x+y}y' &= \frac{\cosh x}{\cosh^2 x} + y^2 + 2xyy' \\
\frac{1}{x+y}y' - 2xyy' &= \frac{1}{\cosh x} + y^2 - \frac{1}{x+y} \\
y' &= \frac{\frac{1}{\cosh x} + y^2 - \frac{1}{x+y}}{\frac{1}{x+y} - 2xy} \\
&= \boxed{\frac{x+y+y^2(x+y)\cosh x - \cosh x}{(1-2xy(x+y))\cosh x}}
\end{aligned}$$

Don't forget to do problem 2, on the other side of this sheet.

2. (5 pts) When a ball (or you on a bicycle) rolls up the side of a parabolic hill, three things can happen: either it runs out of energy before it gets to the top, and so gets as far as it can and then starts rolling back down; or it has more than enough energy to clear the top of the hill and start rolling down the other side; or it has precisely the amount of energy needed to exactly reach the top. In this last situation, it's a physics fact that: **the velocity of the ball is proportional to the distance to the top of the hill**. Let's say that at time $t = 0$, the distance $f(0)$ from the top of the hill is exactly 2 meters, and that the ball at this time is traveling at one meter per minute. At what time t (in minutes) will the ball be precisely one meter from the top of the hill? (Hint: your answer should not be negative.)

The problem says that the velocity of the object is proportional to the distance: $f'(t) = k\, f(t)$. This implies that the distance is an exponential function: $f(t) = f(0)\, e^{kt}$. Since the ball is rolling towards the top of the hill, the distance is decreasing: k is negative.

We know that $f(0) = 2$, and that $|f'(0)| = 1$, because the problem said the speed. But this means that $f'(0) = -1$. On the other hand, $f'(t) = k\, f(t)$, so $-1 = f'(0) = k\, f(0) = k\, 2$, so $k = -1/2$. Thus:

$$f(t) = 2\, e^{-t/2}$$

The problem asks us to find t such that $f(t) = 1$. Thus $1 = 2e^{-t/2}$, so $1/2 = e^{-t/2}$, so $2 = e^{t/2}$, and $\ln 2 = t/2$. Thus the time when the ball is one meter from the top of the hill is $\boxed{t = 2 \ln 2}$.

3. (bonus) What was your favorite topic from the past week of Calculus (chain rule, implicit differentiation, logarithms, inverse-trigonometric functions, hyperbolic functions, physics exercises, exponential growth, linear approximation, differentials, etc.)? Describe the topic is, and why you like it.